An Equivalent Strain Approach for Magneto-Elastic Couplings

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This paper presents an equivalent strain approach to reduce multiaxial magneto-elastic configurations into 1-D descriptions. It follows similar principles as those used for equivalent stress approaches. The approach can be easily implemented into magneto-elastic constitutive equations using strain as driving mechanical state variable. It is particularly useful to describe 3-D configurations when only 1-D measurements are available to describe the magneto-elastic behavior.

Index Terms— **Coupled constitutive laws, equivalent stress, Helmholtz free energy, magneto-mechanical behavior.**

I. Introduction

MAGNETIC and mechanical behavior are strongly coupled. The mechanical response depends on the magnetization state and, conversely, the magnetic response depends on the mechanical state. This coupling is for instance used in the applications of giant magnetostrictive materials [1]. It is also responsible for the strong effect of stress on the magnetic permeability [2]. With the development of electromagnetic devices subjected to severe mechanical loading, the introduction of magneto-mechanical coupling effects into numerical modeling tools is now required so as to design optimal electromagnetic systems.

One way to describe the effect of stress on the magnetic behavior of magnetic materials is to introduce the concept of equivalent stress [3], [4]. An equivalent stress is a uniaxial stress state that affects the magnetic behavior in a similar manner as the real multiaxial stress state. The idea is to limit the magneto-mechanical characterization of the material to the uniaxial case.[1] This characterization is of course incomplete, but can be obtained experimentally using fairly accessible equipment [5], [6]. For the modeling of a real multiaxial configuration, the stress is computed first. The equivalent stress (scalar) is then calculated from the full stress tensor, possibly combined to material parameters (depending on the chosen formulation for the equivalent stress [4]). This uniaxial equivalent stress is then used to define the magnetic properties from the uniaxial characterization results. Such a modeling strategy using an equivalent stress approach has been applied successfully in finite-element analysis to incorporate the effect of stress on the magnetic permeability [7] or on the hysteresis losses in electrical machines [8].

However, some constitutive models for magneto-elastic behavior prefer the use of strain rather than stress as driving state variable [6], [9]–[11]. It is, therefore, useful to define an equivalent strain—rather than an equivalent stress—for magneto-elastic behavior. This paper shows that the same approach derived for equivalent stresses is also applicable to obtain equivalent strains. An illustration is given for an equivalence based on magnetization derived from a thermodynamic description of magneto-elasticity.

Manuscript received November 15, 2016; revised January 25, 2017; accepted January 31, 2017. Date of publication February 2, 2017; date of current version May 26, 2017. Corresponding author: L. Daniel (e-mail: laurent.daniel@centralesupelec.fr).

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Digital Object Identifier 10.1109/TMAG.2017.2663113

[1]For which the stress is uniaxial and applied in the direction parallel to the magnetic field.

II. Magneto-Elastic Constitutive Equations

The magneto-elastic behavior can be described from the Helmholtz free energy [12]. Let us consider the following expression for the free energy:

$$\rho\psi = \frac{1}{2}\lambda I_1^2 + \mu I_2 + \frac{1}{2}\gamma_4 I_4 + \frac{1}{2}\gamma_5 I_5 + \gamma_{14} I_1 I_4 \qquad (1)$$

where the mechanical and magnetic state variables are the total strain tensor $\boldsymbol{\varepsilon}$ and the magnetic induction $\mathbf{B}$, respectively. The dual variables for strain and magnetic induction are then the stress tensor $\boldsymbol{\sigma}$, and the magnetization $\mathbf{M}$, respectively. I_1, I_2, I_4, and I_5 denote the magneto-elastic invariants

$$I_1 = \mathrm{tr}(\boldsymbol{\varepsilon}) \quad I_2 = \mathrm{tr}(\boldsymbol{\varepsilon}^2)$$
$$I_4 = \mathbf{B}.\mathbf{B} \quad I_5 = \mathbf{B}.\boldsymbol{\varepsilon}\,\mathbf{B}. \qquad (2)$$

The operator tr takes the trace of a second-order tensor. λ and μ are the Lamé coefficients to describe isotropic elastic behavior. γ_4 is a material parameter to describe purely magnetic behavior. γ_5 and γ_{14} are two additional material parameters to describe magneto-elastic effects. They can be related to the magnetostriction strain $\boldsymbol{\varepsilon}^\mu$. Assuming a parabolic stress-independent magnetostrictive behavior [9]

$$\boldsymbol{\varepsilon}^\mu = \frac{3}{2}\alpha\left(\mathbf{B}\otimes\mathbf{B} - \frac{1}{3}\mathbf{B}.\mathbf{B}\,\mathbf{I}_d\right) \qquad (3)$$

where $\mathbf{I}_d$ is the second-order identity tensor, γ_5 and γ_{14} can be expressed as (see the Appendix)

$$\gamma_5 = -6\,\alpha\mu \quad \text{and} \quad \gamma_{14} = \alpha\mu. \qquad (4)$$

The constitutive equations for the material behavior are then obtained by derivating the expression of the free energy (1) with respect to the state variables

$$\boldsymbol{\sigma} = \rho\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}} \quad \text{and} \quad \mathbf{M} = -\rho\frac{\partial\psi}{\partial\mathbf{B}}. \qquad (5)$$

It must be noted that the formulation (1) chosen for the free energy provides a pseudolinear magnetic behavior (apart from

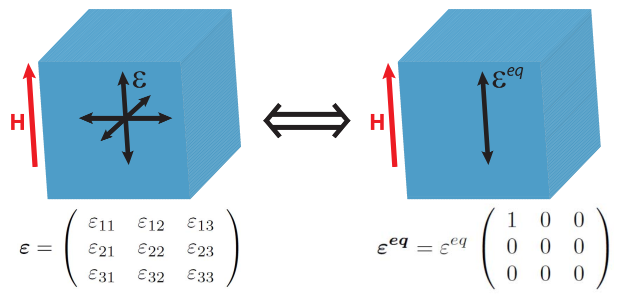

Fig. 1. Principle of the uniaxial equivalent strain. (Left) Real configuration (full strain tensor). (Right) Equivalent strain configuration (uniaxial strain).

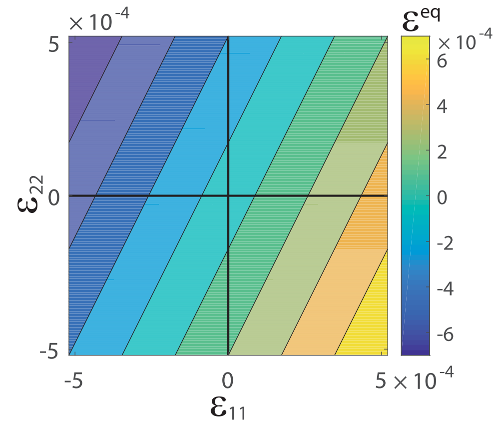

Fig. 2. Equivalent strain for a purely biaxial strain loading. The magnetic field is applied along direction 1.

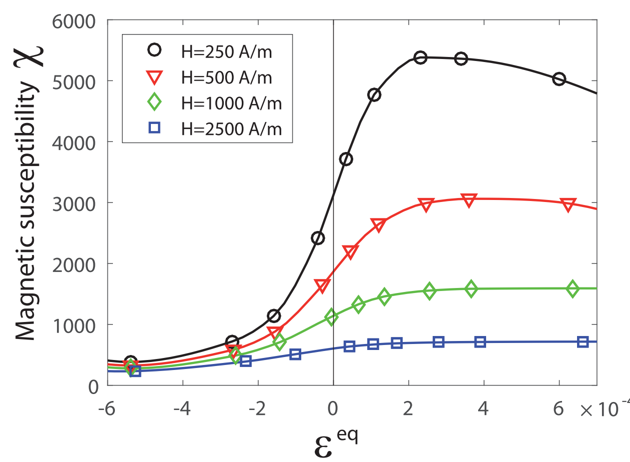

Fig. 3. Experimental results for the magnetic susceptibility under uniaxial magneto-mechanical loading [3]. The x-axis has been transformed into an equivalent strain using (8).

magneto-elastic effects), due to the use of a constant magnetic parameter γ_4. A more accurate description of magnetic nonlinearities would require the use of a more complex function for the free energy [6]. The principle for the derivation of the equivalent strain are, however, unchanged by this simplification.

III. Equivalent Strain

We propose to define an equivalent strain ε^{eq} based on an equivalence in magnetization. Therefore, we search a uniaxial strain, applied parallel to the magnetic field, with norm ε^{eq}, that gives the same magnetization state $\mathbf{M}$ as the real strain state $\boldsymbol{\varepsilon}$ (see Fig. 1).

From (1) and (5), the magnetization $\mathbf{M}$ can be expressed as

$$\mathbf{M} = -\gamma_4\,\mathbf{B} - \gamma_5\,\boldsymbol{\varepsilon}.\mathbf{B} - 2\gamma_{14}\,\mathrm{tr}(\boldsymbol{\varepsilon})\,\mathbf{B}. \tag{6}$$

In the case of a uniaxial strain ε^{eq} applied along the magnetization direction, the magnetization reduces to

$$\mathbf{M} = -\gamma_4\,\mathbf{B} - (\gamma_5 + 2\gamma_{14})\,\varepsilon^{eq}\,\mathbf{B}. \tag{7}$$

Based on an equivalence in magnetization, the equivalent strain ε^{eq} is then obtained by identification of (6) and (7)

$$\varepsilon^{eq} = \frac{\gamma_5\,\varepsilon_{\|} + 2\gamma_{14}\,\mathrm{tr}(\boldsymbol{\varepsilon})}{\gamma_5 + 2\gamma_{14}} \tag{8}$$

where $\varepsilon_{\|}$ is the component of the strain tensor along the magnetization direction ($\varepsilon_{\|} = \mathbf{b}.\boldsymbol{\varepsilon}\,\mathbf{b}$, with $\mathbf{b} = \mathbf{B}/\|\mathbf{B}|$). It can be noticed that in the case of a uniaxial strain applied parallel to the magnetic field, the equivalent strain satisfactorily reduces to the strain amplitude. As an illustration, the proposed equivalent strain is shown in Fig. 2 in the case of a purely biaxial strain state with a principal strain along the magnetic field direction. The obtained figure ressembles the trends of the equivalent stress proposed in [4] and [13].

IV. Application to an Iron–Cobalt Alloy

The equivalent strain approach proposed in Section III can be applied to Iron–Cobalt laminations. An extensive experimental characterization of the magneto-elastic properties of this alloy can be found in [3] and [14]. The uniaxial magnetic behavior of the material is shown in Fig. 3, showing the magnetic susceptibility χ as a function of the mechanical loading. A secant definition is used for the (scalar) magnetic susceptibility, calculated as the ratio between the magnetization measured along the magnetic field and the magnetic field magnitude. The measurements have been performed under uniaxial stress, so that the experimental data had to be transformed in terms of equivalent strain using (8). To perform this operation, the strain is divided into the sum of a purely elastic term and a magnetostriction term (see the Appendix). The material parameters [3] are recalled in Table I. Under elastic isotropy assumptions, the Lamé coefficients λ and μ can be obtained from the Young modulus E and Poisson ratio ν using

TABLE I

Material Parameters

Parameter	E	ν	λ	μ	α
Value	235	0.27	109	92	$16\,10^{-6}$
Unit	GPa	-	GPa	GPa	-

$$\lambda = \frac{\nu\,E}{(1+\nu)(1-2\nu)} \qquad \mu = \frac{E}{2(1+\nu)}. \tag{9}$$

From these uniaxial measurements, the magnetic susceptibility can be predicted for any 3-D strain loading: from the 3-D strain tensor, the equivalent strain is first computed

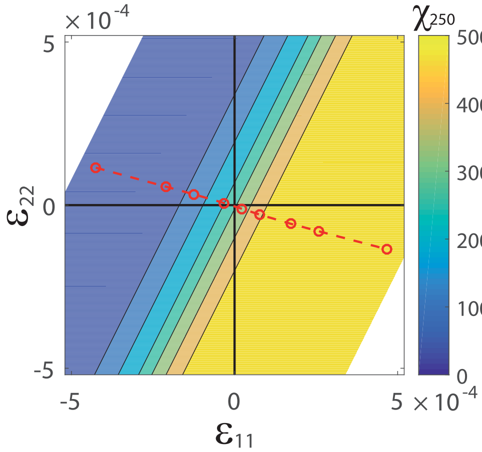

Fig. 4. Prediction of the magnetic susceptibility at H = 250 A/m under biaxial loading using the equivalent strain (8). The data used to describe the uniaxial behavior (Fig. 3) are also reported (circles).

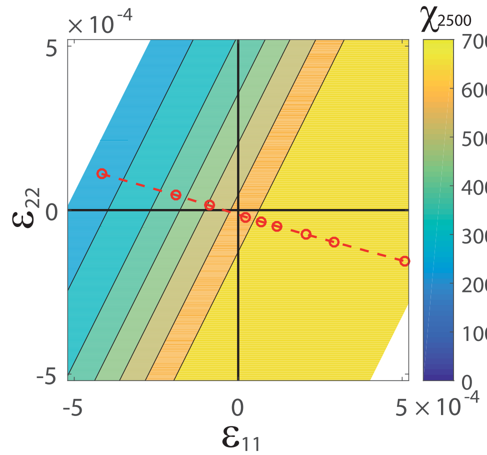

Fig. 5. Prediction of the magnetic susceptibility at H = 2500 A/m under biaxial loading using the equivalent strain (8). The data used to describe the uniaxial behavior (Fig. 3) are also reported (circles).

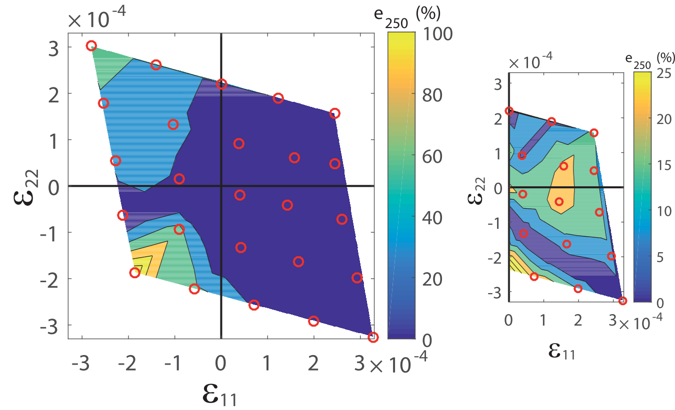

Fig. 6. Error (percent) between the predicted magnetic susceptibility using the equivalent strain (8) and the measured magnetic susceptibility for H = 250 A/m under biaxial mechanical loading [3], [14]. The location of the measurement points in the $(\varepsilon_{11}, \varepsilon_{22})$ plane are also reported (circles). The figure on the right is a focus on positive ε_{11} area with a zoomed-in view scale.

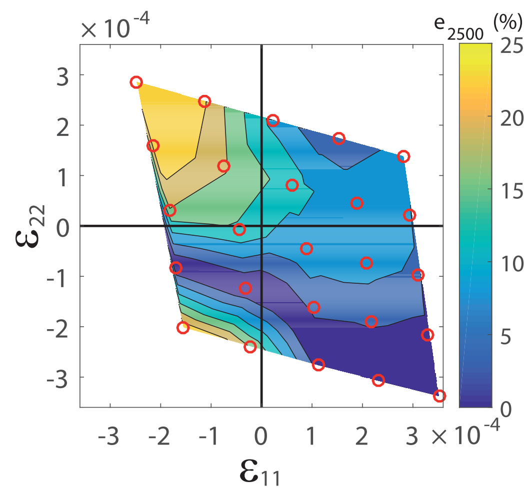

Fig. 7. Error (percent) between the predicted magnetic susceptibility using the equivalent strain (8) and the measured magnetic susceptibility for H = 2500 A/m under biaxial mechanical loading [3], [14]. The location of the measurement points in the $(\varepsilon_{11}, \varepsilon_{22})$ plane is also reported (circles).

using (8), the value of the equivalent strain is then reported in Fig. 3 to obtain the magnetic susceptibility. An illustration for the predicted magnetic susceptibility for H = 250 A/m and H = 2500 A/m is given in Figs. 4 and 5, respectively. The magnetic field is applied along direction 1. The 1-D data used for the prediction have also been reported as circles in the figures, the colormap being extrapolated from these values. These 1-D data are placed along a line with negative slope. This is expected since the experiments have been performed under uniaxal stress applied along the magnetic field direction (index 1). This loading results in a strain along the stress direction combined with a strain of opposite sign along the perpendicular direction due to Poisson effect. The slope is then approximately the Poisson ratio (apart from the magnetostriction strain contribution).

As expected the iso-susceptibility lines follow the isovalues for the equivalent strain in Fig. 2. The modeled values for the magnetic susceptibility are consistent with the fact (observed in Fig. 3) that a tensile stress along the magnetization direction has little effect compared with a compressive stress along the same direction. Since 2-D magneto-mechanical measurements were carried out on this material [3], [14], it is possible to compare the prediction of the equivalent strain approach with the experimental data. The results are shown in Figs. 6 and 7 for H = 250 A/m and H = 2500 A/m, respectively. Circles have been positioned in the figures to show the location of the experimental measurements performed under biaxial mechanical loading. The stress data have been transformed into strain data using the same decomposition of strain into elastic and magnetostrictive part as described for the uniaxial measurements of Fig. 3.

The errors are high when a compression is applied parallel to the magnetization axis, and more particularly for equibi-compression (up to 100% error in the case of H = 250 A/m). The configurations with positive strain component along the applied magnetic field show better accuracy particularly for configurations close to equibitension and pure shear. The errors are lower at high magnetic field (Fig. 7) compared with low magnetic field (Fig. 6) due to the smaller sensitivity of the magnetic susceptibility to mechanical loadings at high field (visible in Fig. 3). The general trends and levels of errors are very similar to those observed for the equivalent stress approaches [3], [4].

V. Conclusion

In this paper, an equivalent strain approach for magneto-elastic behavior has been derived. It can be used in replacement of previous equivalent stress approaches in numerical simulations when the constitutive equations require the use of strain—instead of stress—as the driving mechanical state variable. The results obtained with this equivalent strain approach are very similar in quality to those obtained with standard equivalent stress approaches so that either can be used equally. The choice between stress or strain approaches is then motivated only by the formalism chosen for the constitutive equations describing material behavior. However, both approaches contain rather strong approximations. They provide reasonable trends for the effect of stress on the magnetic behavior, but they cannot replace physical-based constitutive equations if an accurate description of magneto-elastic coupling effects is sought.

Appendix

A. Expression of Coefficients γ_5 and γ_{14}

From the expression (1) of the Helmholtz energy, the stress can be deduced using (5). The calculation yields

$$\boldsymbol{\sigma} = \lambda I_1 \mathbf{I}_d + 2\mu\, \boldsymbol{\varepsilon} + \frac{1}{2}\gamma_5\, \mathbf{B}\otimes\mathbf{B} + \gamma_{14}\mathbf{B}.\mathbf{B}\,\mathbf{I}_d. \quad (10)$$

If we consider a stress-free magnetic loading, the elastic strain is zero (proportional to stress $\boldsymbol{\sigma}$, itself zero) and the total strain $\boldsymbol{\varepsilon}$ is equal to the magnetostriction strain $\boldsymbol{\varepsilon}^\mu$. Equation (10) then becomes

$$\begin{aligned}\boldsymbol{\sigma} &= \lambda\, \mathrm{tr}(\boldsymbol{\varepsilon}^\mu)\,\mathbf{I}_d + 2\mu\, \boldsymbol{\varepsilon}^\mu + \frac{1}{2}\gamma_5\, \mathbf{B}\otimes\mathbf{B} + \gamma_{14}\mathbf{B}.\mathbf{B}\,\mathbf{I}_d \\ &= \mathbf{0}. \end{aligned} \quad (11)$$

Magnetostriction is usually considered as an isochoric deformation so that $\mathrm{tr}(\boldsymbol{\varepsilon}^\mu)$=0. If we apply the parabolic stress-independent magnetostrictive behavior given by (3), the two following relationships between γ_5 and γ_{14} and α and μ are easily obtained:

$$\gamma_5 = -6\,\alpha\mu \qquad \gamma_{14} = \alpha\mu. \quad (12)$$

B. Determination of Strain From Magneto-Mechanical Measurements

The magneto-mechanical measurements [3], [14] used in this paper have been carried out under controlled stress. For the application presented in this paper, it would be more convenient if the total strain had been directly recorded during these experiments. However, the total strain can be retrieved from the stress tensor $\boldsymbol{\sigma}$ using the constitutive law of the material. For that purpose, the total strain $\boldsymbol{\varepsilon}$ is divided into a purely elastic contribution $\boldsymbol{\varepsilon}^e$ given by Hooke's law (13) and a magnetostriction contribution $\boldsymbol{\varepsilon}^\mu$ given by (3)

$$\boldsymbol{\varepsilon}^e = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}\mathrm{tr}(\boldsymbol{\sigma})\,\mathbf{I}_d. \quad (13)$$

To obtain the induction magnitude B required for the calculation of $\boldsymbol{\varepsilon}^\mu$ (3), the measured value χ for the magnetic susceptibility at a given stress and magnetic field H level is used

$$B = \mu_0(1+\chi)\,H. \quad (14)$$

The total strain $\boldsymbol{\varepsilon}$ is then simply defined as

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^\mu. \quad (15)$$

Acknowledgment

The author would like to thank Prof. F. Bouillault, University of Paris-Sud, for useful discussions on thermodynamic approach formalism for magneto-elastic behavior.

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