An Analytical Model for the Effect of Multiaxial Stress on the Magnetic Susceptibility of Ferromagnetic Materials

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The magnetic permeability of magnetic materials highly depends on mechanical stress. Stress state is usually multiaxial and has a significant effect on the performance of electromagnetic devices. In this paper a three-parameter analytical model for the stress dependent permeability of magnetic materials is proposed, based on a simplified energetic description of magneto-elastic behaviour. The proposed approach also provides a new equivalent stress for magnetic behaviour in the low-field and low-stress regime.

***Index Terms*—Effect of stress, equivalent stress, magneto-elastic coupling, magneto-mechanical behaviour, multiaxiality.**

I. Introduction

MAGNETIC behaviour is known to be very sensitive to the application of stress [1], leading to significant effects on the performance of electromagnetic devices. Magnetic constitutive laws including magneto-elastic effects have been proposed in the literature. They often consider uniaxial stress configurations (pure tension or compression) [2]–[5]. When multiaxial magneto-elastic loadings are introduced [6]–[9], the models are often too complex to be easily implemented in structural analysis tools. Approaches based on the use of an equivalent stress can be used [10]–[15] and implemented in numerical simulations [16]. An alternative approach is to implement directly into standard structural analysis tools a magnetic permeability accounting for the effect of multiaxial stress. Such a simplified magneto-elastic model has been proposed for 2D Finite Element calculations [17]. We propose in this paper to extend this approach to define a 3D simplified magneto-elastic constitutive law.

In a first part the simplified 3D magneto-elastic model is detailed. It is shown in the second part how further simplifications can provide a fully analytical definition for the stress-dependent magnetic permeability. The susceptibility under particular loadings is then detailed (Section IV) and a parameter identification method is proposed (Section V). Section VI is dedicated to the model prediction and to the comparison to experimental results. It is finally shown that this analytical approach also provides a new equivalent stress definition for the effect of stress on magnetic behaviour (Section VII).

II. Simplified Magneto-Elastic Model

A magneto-elastic constitutive law can be derived from the description of a magnetic material as a set of magnetic domains with known magnetization (M_s) and random orientation [17]. The local free energy (1) of a magnetic domain k is expressed as the sum of three contributions

$$W_k = W_k^{mag} + W_k^{el} + W_k^{an}. \quad (1)$$

The Zeeman energy W_k^{mag} (2) introduces the effect of the applied magnetic field on the equilibrium state. μ_0 is the vacuum permeability. $\mathbf{H}_k$ and $\mathbf{M}_k$ are the magnetic field and magnetization in the magnetic domain

$$W_k^{mag} = -\mu_0 \mathbf{H}_k . \mathbf{M}_k. \quad (2)$$

The elastic energy W_k^{el} (3) introduces the effect of stress on the magnetic equilibrium. $\boldsymbol{\sigma}$ is the applied stress and $\boldsymbol{\varepsilon}_k^{\mu}$ is the magnetostriction strain in the magnetic domain

$$W_k^{el} = -\boldsymbol{\sigma} : \boldsymbol{\varepsilon}_k^{\mu}. \quad (3)$$

The macroscopic anisotropy can result—for example—from the combination of crystalline anisotropy and crystallographic texture. This macroscopic anisotropy can be described through an anisotropy energy term W_k^{an} added to the free Energy. Eq. (4) gives this additional term for a uniaxial anisotropy along direction $\mathbf{v}$, K being a constant to be identified. If we assume macroscopic isotropy, this term vanishes

$$W_k^{an} = K(\mathbf{u}_k . \mathbf{v})^2. \quad (4)$$

Such an approach, very close to Armstrong model [7], was proposed in [17] in the 2D case. For a 3-dimensional configuration and considering isotropic[1] and isochoric magnetostriction, the following 3D definitions for the local magnetization $\mathbf{M}_k$ and magnetostriction strain $\boldsymbol{\varepsilon}_k^{\mu}$ are used

$$\mathbf{M}_k = M_s \mathbf{u}_k \quad (5)$$

$$\boldsymbol{\varepsilon}_k^{\mu} = \lambda_s \left(\frac{3}{2} \mathbf{u}_k \otimes \mathbf{u}_k - \frac{1}{2} \mathbf{I} \right). \quad (6)$$

M_s is the saturation magnetization of the material, $\mathbf{u}_k$ is the orientation of the magnetization in the domain k, λ_s is the saturation magnetostriction constant and $\mathbf{I}$ the second order identity tensor.

[1]Although significantly anisotropic even in supposedly isotropic materials (see for instance [18]), magnetostriction is often considered isotropic in macroscopic models. However, anisotropic definitions could also be used (see for instance [17]) to the price of additional material parameters.

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The magneto-elastic behaviour is obtained by defining the volume fraction f_k of a domain with orientation $\mathbf{u}_k$ through the use of a Boltzmann probability function [6]

$$f_k = \frac{\exp(-A_s W_k)}{\int_k \exp(-A_s W_k)}. \tag{7}$$

A_s is a material parameter linked to the initial anhysteretic susceptibility χ^o [8]

$$A_s = \frac{3\chi^o}{\mu_0 M_s^2}. \tag{8}$$

Once the probability f_k is defined, the macroscopic magnetization $\mathbf{M}$ and magnetostriction $\boldsymbol{\varepsilon}^\mu$ are obtained thanks to an averaging operation over all possible directions

$$\mathbf{M} = \langle \mathbf{M}_k \rangle = \int_k f_k \mathbf{M}_k \tag{9}$$

$$\boldsymbol{\varepsilon}^\mu = \langle \boldsymbol{\varepsilon}_k^\mu \rangle = \int_k f_k \boldsymbol{\varepsilon}_k^\mu. \tag{10}$$

This integration step can be performed numerically using a discretisation of possible orientations $\mathbf{u}_k$ [9].

Although quite simple, this approach is not analytical due to the integral operation required in (7), (9) and (10). It can be made analytical by considering a further simplified configuration with a limited number of domains, as already suggested in [15] for the definition of an equivalent stress for magnetic behaviour or in [19] to describe the effect of plasticity on the magnetic behaviour.

III. Stress-Dependent Magnetic Susceptibility

We consider a homogeneous isotropic magnetic material submitted to a magnetic field in the direction $\mathbf{x}$ ($\mathbf{H} = H\mathbf{x}$) in an orthonormal coordinate system $(O, \mathbf{x}, \mathbf{y}, \mathbf{z})$. This material is simultaneously submitted to a multiaxial stress state $\boldsymbol{\sigma}$ given as

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{pmatrix}_{xyz}. \tag{11}$$

No assumption is made on the relative orientation between the applied magnetic field and the principal stress directions. Following the approach proposed in [15], the material equilibrium state can be defined using a simplified energy description. The material is assumed to be divided into six domains noted k ($k = \{1, 2, 3, 4, 5, 6\}$) with magnetization oriented along $\mathbf{u}_k$ ($\mathbf{u}_k = \{\mathbf{x}, -\mathbf{x}, \mathbf{y}, -\mathbf{y}, \mathbf{z}, -\mathbf{z}\}$). Each domain k is characterized by its free energy (12) obtained after simplification of (1). For strongly anisotropic media such as textured materials or epitaxial thin films, an anisotropy term (e.g. (4)) should be kept in the free energy

$$W_k = -\mu_o H M_s \mathbf{x}.\mathbf{u}_k - \boldsymbol{\sigma} : \boldsymbol{\varepsilon}_k^\mu. \tag{12}$$

Under these assumptions the free energy for each domain k can be explicitly written. The volume fraction f_k of each domain (7) is then estimated thanks to a discrete summation

$$f_k = \frac{\exp(-A_s W_k)}{\sum_{k=1}^{6} \exp(-A_s W_k)}. \tag{13}$$

The material magnetization $\mathbf{M}$ is also obtained by a discrete summation

$$\mathbf{M} = \sum_{k=1}^{6} f_k M_s \mathbf{u}_k. \tag{14}$$

The magnetization is thus defined analytically. The calculation leads to the following:

$$\mathbf{M} = \frac{\mathrm{A}_x \sinh(\kappa H)}{\mathrm{A}_x \cosh(\kappa H) + \mathrm{A}_y + \mathrm{A}_z} M_s \mathbf{x} \tag{15}$$

with

$$\mathrm{A}_i = \exp(\alpha \sigma_{ii}), \quad i = \{x, y, z\}. \tag{16}$$

This expression introduces the saturation magnetization M_s of the material and two additional constants α ($\alpha = (3/2) A_s \lambda_s$) and κ ($\kappa = \mu_o A_s M_s$). Due to the form of the magnetostriction tensor (6), only the components σ_{xx}, σ_{yy} and σ_{zz} appear in the analytical expression of the magnetization (15).

The tangent magnetic susceptibility χ_x^t (17) is then obtained by derivation of M with respect to the magnetic field H

$$\chi_x^t(H, \boldsymbol{\sigma}) = M_s \kappa \mathrm{A}_x \frac{\mathrm{A}_x + (\mathrm{A}_y + \mathrm{A}_z)\cosh(\kappa H)}{(\mathrm{A}_x \cosh(\kappa H) + \mathrm{A}_y + \mathrm{A}_z)^2}. \tag{17}$$

If the secant magnetic susceptibility χ_x^s (18) is preferred, it can be obtained directly through the ratio M/H

$$\chi_x^s(H, \boldsymbol{\sigma}) = \frac{\mathrm{A}_x \sinh(\kappa H)}{\mathrm{A}_x \cosh(\kappa H) + \mathrm{A}_y + \mathrm{A}_z} \frac{M_s}{H}. \tag{18}$$

Under the considered assumptions (isotropic homogeneous material), (17) and (18) provide the magnetic susceptibility of the material for any multiaxial stress state with any orientation with respect to the magnetic field. Only three material parameters are introduced: the saturation magnetization M_s and two additional parameters κ and α.

IV. Particular Configurations

Further simplifications can be obtained if less complex loadings are considered.

1) If no stress is applied, the magnetization curve reduces to

$$\mathbf{M} = \frac{\sinh(\kappa H)}{2 + \cosh(\kappa H)} M_s \mathbf{x}. \tag{19}$$

2) If no magnetic field is applied, (17) reduces to (20) defining the initial magnetic susceptibility. It can be noticed that

the same expression is obtained from (18) ($\chi_x^s(0,\boldsymbol{\sigma}) = \chi_x^t(0,\boldsymbol{\sigma}) = \chi_x^o(0,\boldsymbol{\sigma})$)

$$\chi_x^o(0,\boldsymbol{\sigma}) = \frac{M_s \kappa \mathrm{A}_x}{(\mathrm{A}_x + \mathrm{A}_y + \mathrm{A}_z)}. \tag{20}$$

3) The initial susceptibility χ_σ^o under uniaxial stress is given by

$$\chi_\sigma^o = \frac{\kappa M_s}{1 + 2\exp(-\alpha \sigma_{xx})}. \tag{21}$$

4) The initial susceptibility χ_o^o under no applied stress is given by

$$\chi_o^o = \frac{1}{3}\kappa M_s. \tag{22}$$

5) If we consider a uniaxial stress σ_{xx} in the direction $\mathbf{x}$, (17) reduces to (23) and (18) reduces to

$$\chi_x^t(H,\sigma_{xx}) = M_s \kappa \mathrm{A}_x \frac{\mathrm{A}_x + 2\cosh(\kappa H)}{(2 + \mathrm{A}_x \cosh(\kappa H))^2} \tag{23}$$

$$\chi_x^s(H,\sigma_{xx}) = \frac{\mathrm{A}_x \sinh(\kappa H)}{2 + \mathrm{A}_x \cosh(\kappa H)} \frac{M_s}{H}. \tag{24}$$

V. Identification of Material Parameters κ and α

The proposed model is based on three material parameters (M_s, κ and α). The identification of the saturation magnetization M_s is standard. The particular configurations of Section IV can be used to identify the model parameters κ and α. For instance, the parameter κ can be identified from the particular case of the initial susceptibility χ_o^o under no applied stress (22). The parameter α can be identified from the effect of an uniaxial stress on the initial slope χ_σ^o of the magnetization curve. Such a choice leads to the definition of the parameters given by

$$\kappa = \frac{3\chi_o^o}{M_s} \tag{25}$$

$$\alpha = \frac{3}{2} p \chi_o^o \tag{26}$$

with

$$p = \left.\frac{\partial \chi_\sigma^o}{\partial \sigma_{xx}}\right|_{\sigma_{xx}=0}. \tag{27}$$

VI. Model Prediction

Very few measurements are available in the literature for validation under complex loadings. The model has been compared to secant permeability measurements obtained in [20] for an iron-cobalt alloy. The parameters have been identified as $M_s = 1.8\ 10^6$ A/m, $\kappa = 4\ 10^{-3}$ m/A and $\alpha = 10^{-7}$ Pa^{-1}.

Considering the simplicity of the model, Fig. 1 shows a satisfactory agreement. This result could probably be enhanced by introducing additional material parameters. The choice here is to keep the simplicity of the model to allow an easy implementation into numerical tools.

This three-parameter analytical model allows the prediction of the magnetization curve under uniaxial stress applied along the direction of the magnetic field (Fig. 2) but also under more complex loadings. For instance Fig. 3 shows magnetization curves under equibiaxial stress.[2]

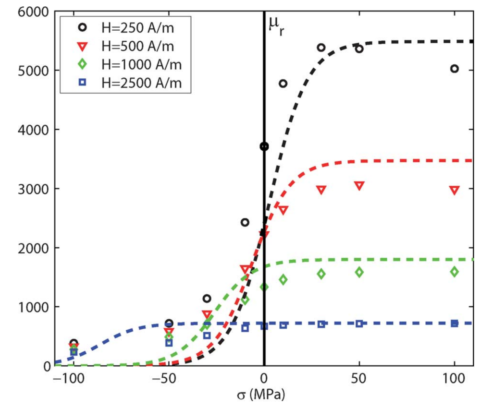

Fig. 1. Relative permeability of an iron-cobalt alloy under uniaxial stress: analytical model (lines) and experimental results [20] (markers).

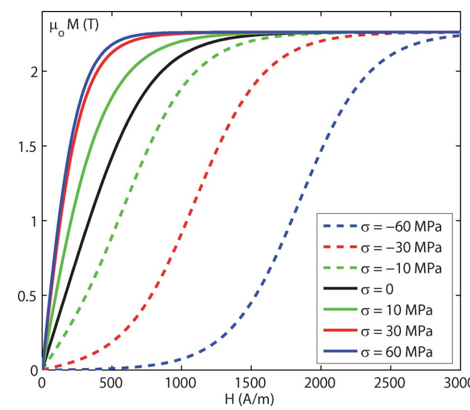

Fig. 2. Magnetization curve under uniaxial stress parallel to the magnetic field ($M_s = 1.8\ 10^6$ A/m, $\kappa = 4\ 10^{-3}$ m/A, $\alpha = 10^{-7}$ Pa^{-1}).

Fig. 4 shows the evolution of the initial susceptibility—defined by (20)—as a function of stress under uniaxial, equibiaxial, hydrostatic and pure shear stress states.[3] It is worth noticing that an applied hydrostatic stress has no effect on the magnetic susceptibility, which is consistent with the fact that magnetostriction strain is isochoric ($\mathrm{trace}(\boldsymbol{\varepsilon}^\mu) = 0$).

VII. Equivalent Stress Definition Based on the Analytical Model

The analytical model can also serve as a basis for the definition of an equivalent stress for magnetic behaviour different from previous proposals [10]–[15]. An equivalence with respect to the initial susceptibility can be defined by the equality of (20) and (21)

$$\frac{\kappa M_s}{1 + 2\exp(-\alpha \sigma_{eq})} = \frac{M_s \kappa \mathrm{A}_x}{(\mathrm{A}_x + \mathrm{A}_y + \mathrm{A}_z)}. \tag{28}$$

This leads to the equivalent stress $\sigma_{eq}^{\chi^o}$ (29). $\sigma_{eq}^{\chi^o}$ is the stress applied in the $\mathbf{x}$ direction that has the same effect on the initial

[2] The stress tensor is then diagonal with the values $(\sigma,\sigma,0)$ on the diagonal, the magnetic field being along the direction $\mathbf{x}$.

[3] Respectively defined by a diagonal stress tensor with the values $(\sigma,0,0)$, $(\sigma,\sigma,0)$, (σ,σ,σ) and $(\sigma,-\sigma,0)$ on the diagonal, the magnetic field being along the direction $\mathbf{x}$.

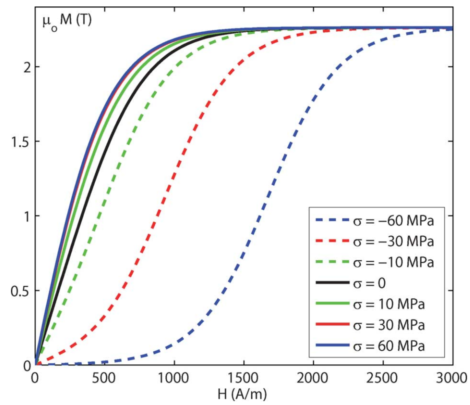

Fig. 3. Magnetization curve under equibiaxial stress with a principal stress parallel to the magnetic field ($M_s = 1.8\ 10^6\ \mathrm{A/m}$, $\kappa = 4\ 10^{-3}\ \mathrm{m/A}$, $\alpha = 10^{-7}\ \mathrm{Pa}^{-1}$).

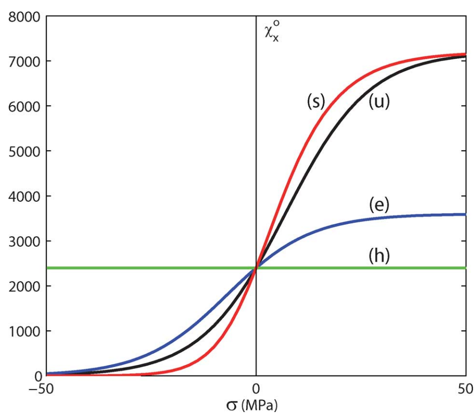

Fig. 4. Initial magnetic susceptibility as a function of stress intensity under uniaxial (u), equibiaxial (e), hydrostatic (h) and pure shear (s) stress states ($M_s = 1.8\ 10^6\ \mathrm{A/m}$, $\kappa = 4\ 10^{-3}\ \mathrm{m/A}$, $\alpha = 10^{-7}\ \mathrm{Pa}^{-1}$).

susceptibility than the multiaxial stress $\boldsymbol{\sigma}$ that defines A_x, A_y and A_z

$$\sigma_{eq}^{\chi^o} = \frac{1}{\alpha} \ln\left(\frac{2\mathrm{A}_x}{\mathrm{A}_y + \mathrm{A}_z}\right). \tag{29}$$

It can be noticed that this equivalent stress is material dependent through the use of parameter α, identified from (26). It can also be noticed from Section III that α is proportional to the saturation magnetostriction strain λ_s and to the initial anhysteretic permeability χ_0 under no applied stress. α is also inversely proportional to the square of the saturation magnetization M_s

$$\alpha = \frac{9\chi^o \lambda_s}{2\mu_0 M_s^2}. \tag{30}$$

$\sigma_{eq}^{\chi^o}$ does not depend on the applied magnetic field, but due to its definition, it can be expected that it is only valid for low applied magnetic fields. More complex definitions can be obtained if we consider the full expressions of the tangent (17) and secant (18) susceptibility. These definitions will generally depend on the level of the magnetic field.

VIII. Conclusion

In this paper an analytical model for the definition of the stress dependent susceptibility of magnetic materials has been proposed. It is based on a very simplified description of the energetic equilibrium underlying magnetic behaviour. The multiaxiality of stress is naturally introduced in this approach and no assumption is made on the relative orientation between stress and magnetic field. This analytical model can be readily implemented in standard structural analysis tools. Due to the strong assumptions made in the construction of the model, it has to be used with caution, particularly if the material is strongly anisotropic, strongly heterogeneous, or if high level of stress (or low magneto-crystalline anisotropy) induces magnetization rotation. The proposed approach also provides a new equivalent stress for magnetic behaviour at low magnetic field.

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