

The concept of equivalent stress and its application to magneto-elastic behaviour

Abstract — Computational magnetics tools are increasingly required to account for the effect of stress on magnetic behaviour. However, not all simulation software is designed for the implementation of magneto-mechanical effects. Using an equivalent stress approach is probably the most accessible option for straightforward incorporation of magneto-elastic coupling effects into electromagnetic computations. Starting from the Von Mises equivalent stress used in the mechanics of plasticity, this article details how a similar concept can be applied to derive magneto-elastic equivalent stresses. Guidelines for the practical implementation of equivalent stress approaches are then provided, and an illustrative example is presented to compare the various approaches proposed in the literature.

I. INTRODUCTION

The magnetic response of a ferromagnetic material to an external stimulus is highly dependent on its mechanical state (Bozorth, 1951; Chikazumi, 1997; Cullity and Graham, 2009). This is mainly due to the fact that ferromagnetic materials are also ferroelastic. Stress being a second order tensor and the magnetic field a vector, it is very challenging to describe the magneto-elastic coupling effects for fully multi-axial configurations.

There are very few experimental setups available to explore the effect of multi-axial stress states on the magnetic response of ferromagnetic materials (Kai et al., 2011, 2014; Rekik et al., 2014b; Aydin et al., 2019a; Zhang et al., 2022; Huang et al., 2024; Li et al., 2024; Le Soudeur and Hubert, 2025). In contrast, numerous experimental setups are available for characterising the magnetic behaviour under uniaxial stress (a comprehensive list would be too long but see for instance Singh et al. (2015); Leuning et al. (2016); Perevertov (2017); Domenjoud et al. (2019); Mailhé et al. (2020); Zhu et al. (2024)). This means that the computational electromagnetics engineers have to deal with multi-axial configurations while only having access to 1D measurements.

The development of fully three-dimensional magneto-mechanical models is a desirable and active line of research to overcome this issue. While this research continues, another modelling option is explored here. The question addressed in this article is *how to transform a 3D magneto-elastic configuration into an equivalent 1D problem that can be addressed with the classical 1D magneto-mechanical characterisation data?*

The heritage of solid mechanics for the study of metal plasticity can be a stimulating inspiration for the definition of equivalent stress approaches suited to magneto-elastic problems.

In a first part, the Von Mises equivalent stress is introduced. Various concepts of equivalent stress for magneto-elastic behaviour are then reviewed. An illustrative example for the practical implementation of equivalent magneto-elastic approaches is finally detailed to compare the available approaches.

II. THE INSPIRATIONAL VON MISES STRESS FOR THE PLASTICITY OF METALS

After an introduction of the required notations, this section details how the Von Mises stress is obtained from an energy equiv-

alence, how it is used as a plasticity criterion and why it is not adapted for magneto-elastic problems.

A. ELASTICITY, PLASTICITY AND THE YIELD STRESS

When subjected to a stress state, materials deform. The stress state is described by a second order tensor σ . The stress tensor is usually symmetric. Expressed in a coordinate system (x, y, z) , it can be expressed as:

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix}_{xyz} \quad (1)$$

There is a coordinate system (X, Y, Z) , the principal coordinate system, in which the expression of the stress tensor is diagonal. The stress tensor is then expressed as:

$$\sigma = \begin{pmatrix} \sigma_I & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & \sigma_{III} \end{pmatrix}_{XYZ} \quad (2)$$

σ_I , σ_{II} and σ_{III} are called the principal stresses, or eigenstresses.

The dual state variable for stress is the strain ϵ , that describes the deformation of the matter. As long as the stress does not reach a critical intensity, the mechanical behaviour can be described by an elastic constitutive equation. The stress σ is associated to an elastic strain ϵ_{el} . When the stress is released, the material reversibly returns to its initial state. If the stress reaches a critical intensity, the material exits the elastic regime, and the behaviour becomes irreversible. In metals, this is due to the ignition of plasticity mechanisms: a plastic strain ϵ_p is superimposed to the elastic strain (under small strain assumption: $\epsilon = \epsilon_{el} + \epsilon_p$), and when the stress is released, this plastic strain remains as a permanent deformation of the material under test.

In the case of linear elastic behaviour, the behaviour can be described by Hooke's law. For an isotropic material, the constitutive equation reads:

$$\sigma = 2\mu \epsilon_{el} + \lambda \text{tr}(\epsilon_{el}), \quad (3)$$

where λ and μ are the Lamé coefficients (material elastic constants).

A standard test to characterise the mechanical behaviour is the tensile test. A uniaxial stress with an amplitude σ_u is applied to the material. The yield stress σ_Y is the value of applied tensile stress σ_u below which the mechanical behaviour is reversible (elastic). If the amplitude σ_u exceeds σ_Y , the behaviour becomes irreversible. σ_Y is a very standard material parameter in mechanical engineering.

In many applications, it is important for engineers to verify that operating stresses do not reach a magnitude that would generate irreversible strains. The stress, however, is generally multi-axial, and there is not trivial way to know if a given stress state σ is associated to the elastic regime or if it triggers plasticity.

The objective of a plasticity criterion is precisely to define a level of stress intensity from which plasticity mechanisms are triggered. The Von Mises criterion is a plasticity criterion that can be interpreted as a maximum distortion strain energy criterion (Hencky, 1924; Hill, 1950).

B. DISTORSION STRAIN ENERGY

The elastic energy W_{el} stored in a material under a stress state σ associated to an elastic strain ϵ_{el} is given by:

$$W_{el} = \frac{1}{2} \sigma : \epsilon_{el} \quad (4)$$

Decomposing stress (resp. strain) into spherical s (resp. ϵ^s) and deviatoric d (resp. ϵ^d) parts¹, and noting that the double-dot product² of a spherical tensor with a deviatoric tensor is zero, the elastic energy reads:

$$\begin{aligned} W_{el} &= \frac{1}{2} (s + d) : (\epsilon^s + \epsilon^d) \\ &= \frac{1}{2} s : \epsilon^s + \frac{1}{2} d : \epsilon^d \end{aligned} \quad (5)$$

The elastic energy is hence decomposed into a spherical part $W_{el}^s = \frac{1}{2} s : \epsilon^s$ and a deviatoric part $W_{el}^d = \frac{1}{2} d : \epsilon^d$. The first term describes the dilatation part of the deformation while the second term describes the distortion part of the deformation. W_{el}^d is called the distortion energy.

Using the Hooke law for an isotropic material (3), it is easy to show that $d = 2\mu \epsilon^d$, so that the distortion energy can be written:

$$W_{el}^d = \frac{1}{4\mu} d : d \quad (6)$$

C. CASE OF A UNIAXIAL TENSILE TEST

Lets consider a uniaxial stress state σ_u of magnitude σ_u along the direction u : $\sigma_u = \sigma_u u \otimes u$, with $\otimes$ the cross-product³. The

corresponding distortion energy W_u^d , derived from the general expression (6) is defined as:

$$W_u^d = \frac{\sigma_u^2}{6\mu} \quad (7)$$

For a tensile stress state, the materials remains in the elastic regime as long as σ_u remains below σ_Y . In other words the behaviour is reversible as long as W_{el}^d remains below a critical value W_{crit}^d :

$$W_{crit}^d = \frac{\sigma_Y^2}{6\mu} \quad (8)$$

D. MAXIMUM DISTORTION STRAIN ENERGY CRITERION

The Von Mises criterion assumes that, for any stress state, the material remains in the elastic regime if the distortion energy W_{el}^d (6) does not exceed the critical value W_{crit}^d (8) (Hencky, 1924; Hill, 1950). The criterion can be expressed as:

$$\text{Elastic regime as long as } \frac{1}{4\mu} d : d \leq \frac{\sigma_Y^2}{6\mu}, \quad (9)$$

that can be reformulated into:

$$\text{Elastic regime as long as } \sqrt{\frac{3}{2}} d : d \leq \sigma_Y. \quad (10)$$

The expression on the left-hand side is called the Von Mises stress and noted σ_{eq}^{VM} (11).

$$\sigma_{eq}^{VM} = \sqrt{\frac{3}{2}} d : d \quad (11)$$

It is a positive scalar, and it is insensitive to the stress tensor orientation. It can be kept in its compact form (11), expressed as a function of the components of the stress tensor (12) or as a function of the principal stresses (13).

$$\sigma_{eq}^{VM} = \sqrt{\frac{1}{2} [(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\sigma_{23}^2 + \sigma_{31}^2 + \sigma_{12}^2)]} \quad (12)$$

$$\sigma_{eq}^{VM} = \sqrt{\frac{1}{2} [(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2]} \quad (13)$$

E. PRACTICAL USE OF VON MISES STRESS

The application of the Von Mises criterion is relevant to anticipate the appearance of plasticity in isotropic metallic materials. The practical implementation is as follows. For a given structure under prescribed loading and boundary conditions, the stress distribution can be determined based on elasticity assumptions. The results is a map of the stress σ (six components σ_{ij}) at each point in the structure. At each point P , the Von Mises stress $\sigma_{eq}^{VM}(P)$ can be calculated using (11), and compared to the yield stress σ_Y of the material. If σ_{eq}^{VM} remains below σ_Y at any point P , no plastic deformation occurs, and the elasticity as-

sumptions are valid. If not, the criterion is not fulfilled, meaning that plastic strain appears in the structure, so that the elasticity assumptions cannot be applied. Fig.1 gives an example of the Von Mises stress distribution resulting from the shrink-fitting of the rotor (inner edge) and stator (outer edge) of a permanent magnet synchronous machine (Roppert et al., 2025). The typical yield stress of Iron-Silicon steels is typically above 300 MPa (Bozorth, 1951; Backes et al., 2024), so it can be concluded from Fig.1 that there is no risk of plastic strain under the considered loading conditions.

F. VON MISES STRESS: A WARNING

Because of its relevance to the plasticity of metals, and its straightforward implementation, Von Mises stress is embedded in most simulation tools dealing with mechanical equations, and notably most finite element software. The concept of equivalent stress is appealing also in the context of magnetic behaviour, and

¹The spherical part A^s of a second order tensor A is defined as $A^s = \frac{1}{3} \text{tr}(A) \mathbf{I}$, where $\text{tr}(\cdot)$ is the trace operator, and $\mathbf{I}$ the second order Identity tensor. The deviatoric part A^d of A is then $A^d = A - A^s = A - \frac{1}{3} \text{tr}(A) \mathbf{I}$. It is easy to show that $\text{tr}(A^d) = 0$.

²The double-dot product of two second order tensors A and B is defined as $A : B = A_{ij} B_{ij}$, using the Einstein summation convention.

³ $a \otimes b = a_i b_j$.

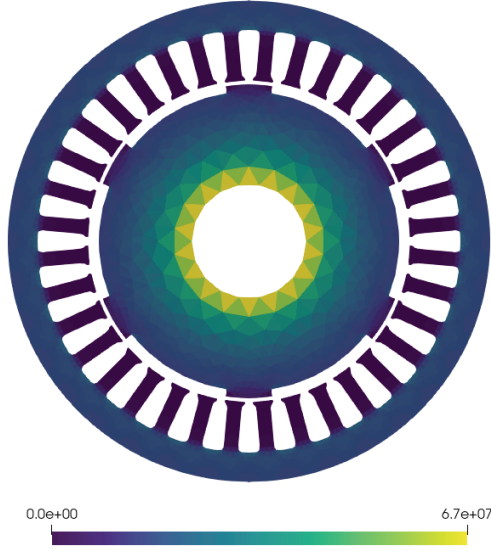


Fig. 1. Von Mises stress (in Pa) after prescribing a purely radial displacement at the rotor's inner and stator's outer edge. Figure taken from Roppert et al. (2025).

some authors have been tempted to apply the Von Mises stress in the context of magneto-elasticity. This temptation, however, should be resisted.

First, the temptation is understandable: the magnetic behaviour is highly sensitive to stress. The stress state in most magnetic devices is multiaxial, but the available magneto-elastic models are mostly uniaxial, and the available magneto-elastic characterisation is performed under uniaxial loadings. It is therefore understandable, when confronted to a multiaxial stress state, to seek a - fictitious - uniaxial stress state that would be *equivalent* to the - actual - stress state. This equivalent stress would allow to refer to the corresponding uniaxial models or uniaxial characterisation setups.

But using the Von Mises stress is misleading, and, in some occasions, can be even worse than just ignoring the effect of stress on the magnetic behaviour. A few illustrative examples can be given. For instance, the Von Mises stress is always positive, because it assumes that the compressive yield stress is identical to the tensile yield stress. In the magneto-elastic context, however, assuming that the effect of a compressive stress is identical to that of a tensile stress contradicts experimental evidence (see for instance (LoBue et al., 2000; Permiakov et al., 2004; Singh et al., 2015; Karthaus et al., 2019; Ouazib et al., 2024)). Another example is that Von Mises stress, as a purely mechanical criterion, ignores the relative orientation between the applied stress and the magnetic field. But an identical stress state can be beneficial or detrimental to the magnetic response depending on the orientation of the magnetic field (see for instance (Kai et al., 2011; Rekik et al., 2014b; Aydin et al., 2019a,b)).

As a conclusion, the Von Mises equivalent stress is a very powerful tool in the context of the plasticity of metals, and has long proven its utility. However, it was never designed to account for magneto-mechanical behaviour. Dedicated equivalent stress approaches must therefore be developed for this specific context.

III. EQUIVALENT STRESSES FOR MAGNETIC MATERIALS

Several research groups have worked to define a concept of equivalent stress suited to the magneto-elastic behaviour. The following is a review of different attempts.

A. THE PIONEERS

The first approaches considered a particular 2D-case with $\sigma_{III} = 0$ and the magnetic field $\mathbf{H}$ aligned along the direction of the principal stress σ_I (see illustration in Fig 2).

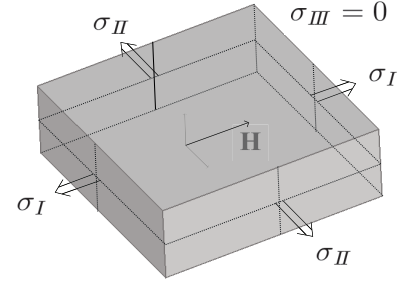


Fig. 2. Canonical 2D magneto-mechanical loading configuration considered in the first equivalent stress approaches.

Schneider and Richardson (1982) proposed the following expression for the equivalent stress σ_{eq}^{SR} :

$$\sigma_{eq}^{SR} = \sigma_I - \sigma_{II} \quad (14)$$

It is remarkable that for an equibiaxial stress, σ_{eq}^{SR} is zero, assuming that such a stress state has no effect on the magnetic behaviour.

Kashiwaya (1991) considered a similar 2D-case but defined a different equivalent stress σ_{eq}^K :

$$\sigma_{eq}^K = k (\sigma_I - \max(\sigma_I, \sigma_{II})) \quad (15)$$

where k is a constant that was chosen to adjust experimental measurements. It can be noted that σ_{eq}^K is always negative or null. If σ_I is the highest principal stress (magnetic field applied along the direction of the maximum principal stress), σ_{eq}^K is zero, so that a tensile stress or an equi-biaxial tension or compression are supposed to have no effect on the magnetic behaviour. It can also be noted that in the case of a uniaxial tension σ_u along the direction of the magnetic field, σ_{eq}^K does not reduce to σ_u .

Still for the same 2D-configuration, Sablik et al. (1994) defined the following equivalent stress σ_{eq}^S :

$$\sigma_{eq}^S = \begin{cases} \frac{1}{3}(2\sigma_I - \sigma_{II}) & \text{for } \sigma_I < 0 \\ \frac{1}{3}(\sigma_I - 2\sigma_{II}) & \text{for } \sigma_I \geq 0 \end{cases} \quad (16)$$

In contrast to the first two proposals, equi-biaxial tension and equi-biaxial compression do not lead to identical values of σ_{eq}^S . But the equivalent stress function is discontinuous in the vicinity of $\sigma_I = 0$, and in the case of a uniaxial tension σ_u along the direction of the magnetic field, σ_{eq}^S does not reduce to σ_u .

Pearson et al. (2000) also proposed an equivalent stress for a biaxial mechanical loading. It takes the form of a polynomial interpolation identified from biaxial measurements. Since the identification requires complex measurements, and is sample dependent, the practical implementation of this approach is not straightforward.

These first seminal approaches were key in highlighting the need for equivalent stress definitions adapted to the study of magneto-mechanical behaviour. However, they suffer strong limitations (Daniel and Hubert, 2010). In particular, these equivalent stresses are not fully multiaxial: they are restricted to 2D-configurations, but even more, they impose that the magnetic

field is parallel to a principal stress direction. Such a constraint is usually not fulfilled in practical applications. This makes the range of application of these approaches very narrow. These limitations have motivated the quest for more general approaches.

B. PRINCIPAL STRESS DECOMPOSITION (PSD) METHOD

The principal stress decomposition (PSD) method initially proposed by Nakano et al. (2009) is not strictly-speaking an equivalent stress approach, but it has the same goal of being able to describe 3D configurations from 1D information only. The approach was proposed for 2D magneto-mechanical loadings, but it can be generalised to 3D without significant obstacle.

The principle is as follows. The stress is expressed in the principal coordinate system (X, Y) . In that coordinate system the magnetic field $\mathbf{H}$ is defined by its components H_X and H_Y . The assumption of the PSD method is that the magnetic induction $\mathbf{B}$ can be decomposed into its two components B_X and B_Y , where B_X is only influenced by σ_I and B_Y is only influenced by σ_{II} :

$$B_X = B_X(H_X, \sigma_I) \text{ and } B_Y = B_Y(H_Y, \sigma_{II}) \quad (17)$$

A strong limitation with the PSD method is that the magnetic response in one direction is totally independent of the stress in the perpendicular direction. This contradicts experimental observation showing that a uniaxial tension stress in a given direction has an effect on the permeability in the perpendicular direction, this effect resembling that of a compressive stress in that perpendicular direction. Moreover, it is clear that this approach will predict a strong effect of a hydrostatic stress⁴ on the magnetic response, again in contradiction with experimental evidence. These limitations could probably be overcome, at least partially, by using the deviatoric tensor $\mathbf{d}$ instead of the full stress tensor $\boldsymbol{\sigma}$. It seems this path has not been explored so far. We will call this approach the principal deviatoric stress decomposition (PDSD) method. It simply consists in replacing (17) by (18):

$$B_X = B_X(H_X, \frac{3}{2}d_I) \text{ and } B_Y = B_Y(H_Y, \frac{3}{2}d_{II}) \quad (18)$$

The factor $3/2$ has been introduced to enforce that the approach reduces to the 1D behaviour in the case of a uniaxial stress loading applied parallel to the magnetic field.

C. EQUIVALENT STRESS BASED ON AN EQUIVALENCE IN MAGNETO-ELASTIC ENERGY

Following the inspirational work of Von Mises, and its energy interpretation, equivalence in energy can be derived to design equivalent stress definitions for magneto-elastic behaviour.

C.1 Deviatoric magneto-elastic equivalent stress

A first approach (Daniel and Hubert, 2009) relies on the following definition of the magneto-elastic energy $W_{\sigma\mu}$ (Hubert and Schäfer, 1998):

$$W_{\sigma\mu} = -\boldsymbol{\sigma} : \boldsymbol{\varepsilon}_\mu \quad (19)$$

where $\boldsymbol{\sigma}$ is the stress applied to the material, and $\boldsymbol{\varepsilon}_\mu$ the magnetostriction strain tensor. For an isotropic material, the magnetostriction strain tensor can be defined as:

⁴A hydrostatic $\boldsymbol{\sigma}^h$ stress is a purely spherical tensor in the form $\boldsymbol{\sigma}^h = \sigma_o \mathbf{I}$.

$$\boldsymbol{\varepsilon}_\mu = \lambda(\|\mathbf{H}\|) \left(\frac{3}{2} \mathbf{h} \otimes \mathbf{h} - \frac{1}{2} \mathbf{I} \right) \quad (20)$$

where $\lambda(\|\mathbf{H}\|)$ is the longitudinal amplitude of the magnetostriction strain, assumed to depend only on the magnitude of the magnetic field $\mathbf{H}$, $\mathbf{h}$ is the direction (unit vector) of the magnetic field $\mathbf{H}$, and $\mathbf{I}$ is the second-order identity tensor.

Decomposing the stress $\boldsymbol{\sigma}$ into its spherical $\mathbf{s}$ and deviatoric $\mathbf{d}$ parts, the magneto-elastic energy can be re-written:

$$W_{\sigma\mu} = -\lambda(\|\mathbf{H}\|) (\mathbf{s} + \mathbf{d}) : \left(\frac{3}{2} \mathbf{h} \otimes \mathbf{h} - \frac{1}{2} \mathbf{I} \right) \quad (21)$$

$\mathbf{h}$ being a unit vector, $\mathbf{s} : (\mathbf{h} \otimes \mathbf{h}) = \frac{1}{3} \text{tr}(\mathbf{s})$. Moreover, $\mathbf{s} : \mathbf{I} = \text{tr}(\mathbf{s})$ and $\mathbf{d} : \mathbf{I} = 0$. The magneto-elastic energy $W_{\sigma\mu}$ can hence be written:

$$W_{\sigma\mu} = -\frac{3}{2} \lambda(\|\mathbf{H}\|) \mathbf{d} : (\mathbf{h} \otimes \mathbf{h}) \quad (22)$$

$\mathbf{d} : (\mathbf{h} \otimes \mathbf{h})$ is the projection of $\mathbf{d}$ along the direction $\mathbf{h}$, so that the magneto-elastic energy can finally be written:

$$W_{\sigma\mu} = -\frac{3}{2} \lambda(\|\mathbf{H}\|) {}^t\mathbf{h} \cdot \mathbf{d} \cdot \mathbf{h} \quad (23)$$

In the case of a uniaxial stress state $\boldsymbol{\sigma}_u$ applied parallel to the magnetic field ($\boldsymbol{\sigma}_u = \sigma_u \mathbf{u} \otimes \mathbf{u}$ with $\mathbf{u} = \mathbf{h}$), and starting from (21), the magneto-elastic $W_{\sigma\mu}^u$ energy reads:

$$W_{\sigma\mu}^u = -\sigma_u \lambda(\|\mathbf{H}\|) (\mathbf{h} \otimes \mathbf{h}) : \left(\frac{3}{2} \mathbf{h} \otimes \mathbf{h} - \frac{1}{2} \mathbf{I} \right) \quad (24)$$

Noting that, $\mathbf{h}$ being a unit vector, $(\mathbf{h} \otimes \mathbf{h}) : (\mathbf{h} \otimes \mathbf{h}) = 1$ and $(\mathbf{h} \otimes \mathbf{h}) : \mathbf{I} = 1$, it comes:

$$W_{\sigma\mu}^u = -\sigma_u \lambda(\|\mathbf{H}\|) \quad (25)$$

Defining a magneto-elastic energy criterion consists in assuming that two stress states are equivalent, in terms of magneto-elastic behaviour, if they share the same magneto-elastic energy $W_{\sigma\mu}$. When considering a general stress state, with its magneto-elastic energy $W_{\sigma\mu}$ defined by (23), we can look for the uniaxial stress state σ_u applied in the direction $\mathbf{h}$ of the magnetic field, that leads to the same magneto-elastic energy. This comes down to equalising (23) and (25). This is the approach followed in Daniel and Hubert (2009) to define the equivalent stress σ_{eq}^{DH} (26) based on an equivalence in magneto-elastic energy:

$$\sigma_{eq}^{DH} = \frac{3}{2} {}^t\mathbf{h} \cdot \mathbf{d} \cdot \mathbf{h} \quad (26)$$

This expression is fully multiaxial. Any stress tensor $\boldsymbol{\sigma}$ can be considered in combination with any magnetic field direction $\mathbf{h}$. σ_{eq}^{DH} can be positive or negative, and the implementation is straightforward. However, it can be noticed that the approach is restricted to isotropic materials. It can also be noticed that due to the projection along the magnetic field direction $\mathbf{h}$, the shear stress terms along the field direction do not have any effect on the definition of σ_{eq}^{DH} , meaning that it is assumed that they have no effect on the magneto-elastic behaviour.

C.2 Extension to orthotropic materials

Many magnetic materials are anisotropic, and it impacts strongly their magneto-elastic response. Hubert and Daniel (2011) attempted to define an equivalent stress based on an equivalence

in magneto-elastic energy for orthotropic material. They succeeded in obtaining an analytical definition, but the implementation revealed to be complex, because it required to identify the principal coordinate system for magnetostriction strain which, in the case of anisotropic materials, has no reason to be the same as the stress principal coordinate system. Unless the magnetic field is applied along one orthotropy axis of the material, this approach is not easily practicable.

C.3 Shear-dependent equivalent stress

Rasilo et al. (2019) developed a similar energy equivalence, but based on a macroscopic definition for the magneto-elastic potential (Fonteyn et al., 2010; Aydin et al., 2017). The following equivalent stress is obtained:

$$\sigma_{eq}^{Ra} = \begin{cases} r - \sqrt{{}^t\mathbf{h} \cdot \left(r\mathbb{I} - \frac{3}{2}\mathbb{d}\right) \cdot \mathbf{h}}, & \text{if } \sigma_{eq}^{DH} \leq r \\ r + \sqrt{{}^t\mathbf{h} \cdot \left(r\mathbb{I} - \frac{3}{2}\mathbb{d}\right) \cdot \mathbf{h}}, & \text{otherwise} \end{cases} \quad (27)$$

with $\mathbf{t}_1$ and $\mathbf{t}_2$ two orthogonal directions perpendicular to the magnetic field direction $\mathbf{h}$ (unit vectors). K is a material parameter. It was shown that K can be expressed as a function of more standard material parameters:

$$K = \frac{9}{2} \frac{\chi_0 \lambda_m}{\mu_0 M_s^2} \quad (29)$$

χ_0 is the initial susceptibility measured on an anhysteretic curve under no applied stress, λ_m is the maximum longitudinal magnetostriction strain under no applied stress, μ_0 is the vacuum magnetic permeability and M_s is the saturation magnetisation of the material. For isotropic polycrystals, the maximum magnetostriction strain λ_m can analytically be defined from the single crystal properties (Daniel et al., 2008). In the simple case where stress is considered uniform within the polycrystal, the expression of λ_m reduces to:

$$\begin{cases} \lambda_m = \frac{2}{5} \lambda_{100} & \text{for positive magneto-crystalline} \\ & \text{anisotropy materials} \\ \lambda_m = \frac{3}{5} \lambda_{111} & \text{for negative magneto-crystalline} \\ & \text{anisotropy materials} \end{cases} \quad (30)$$

One issue with the definition of σ_{eq}^{HD} (28) is that $\mathbf{t}_1$ and $\mathbf{t}_2$ are not uniquely defined from the knowledge of the magnetic field direction $\mathbf{h}$. Hence, the value of σ_{eq}^{HD} is not uniquely defined for a given magneto-elastic loading. In the case of peculiar geometries, such as electrical sheets, Hubert and Daniel (2011) suggest that $\mathbf{t}_2$ can be taken as the direction normal to the sheet plane. This makes the definition of σ_{eq}^{DH} unique, but is only applicable as long as the magnetic field $\mathbf{H}$ remains in-plane.

A similar approach was followed in Daniel (2013): from an analytical expression $\mathbf{M}(\mathbf{H}, \boldsymbol{\sigma})$ for the magnetisation, and applying

⁵Note that for the consistency of the paper, a slightly different notation is adopted here: the parameter k used by Hubert and Daniel (2011) is equal to $2K/3$.

In contrast to the deviatoric equivalent stress σ_{eq}^{DH} , the expression (27) introduces a material parameter r , defined as the uniaxial stress amplitude providing the maximum permeability for the material, the permeability being measured in the direction parallel to the applied stress. A great advantage of this approach over the other equivalent stresses is that it naturally introduces a dependence to the shear stress along the magnetic field direction $\mathbf{h}$. As noted by Rasilo et al. (2019), when there is no such shear stress component, σ_{eq}^{Ra} reduces to σ_{eq}^{DH} .

D. OTHER EQUIVALENCES

D.1 Equivalence in magnetic susceptibility

As an alternative to the equivalence in magneto-elastic energy, Hubert and Daniel (2011) developed an equivalent stress σ_{eq}^{HD} based on an equivalence in magnetic susceptibility. From a simplified description of the magnetisation process based on a multiscale model, the following expression was obtained⁵:

$$\sigma_{eq}^{HD} = \frac{2}{3} \sigma_{eq}^{DH} - \frac{1}{K} \ln \left[\frac{1}{2} (\exp(K {}^t\mathbf{t}_1 \cdot \mathbb{d} \cdot \mathbf{t}_1) + \exp(K {}^t\mathbf{t}_2 \cdot \mathbb{d} \cdot \mathbf{t}_2)) \right] \quad (28)$$

an equivalence in magnetisation, an equivalent stress σ_{eq}^D was obtained:

$$\sigma_{eq}^D = \frac{1}{K} \ln \left(\frac{2 \exp(K \sigma_{//})}{\exp(K \sigma_{\perp 1}) + \exp(K \sigma_{\perp 2})} \right) \quad (31)$$

K is the same parameter as defined by (29). $\sigma_{//}$ is the projection of the stress tensor $\boldsymbol{\sigma}$ along the magnetic field direction ($\sigma_{//} = {}^t\mathbf{h} \cdot \boldsymbol{\sigma} \cdot \mathbf{h}$). $\sigma_{\perp 1}$ and $\sigma_{\perp 2}$ are the projections of the stress tensor $\boldsymbol{\sigma}$ along two perpendicular directions orthogonal to the magnetic field direction $\mathbf{h}$. Introducing in (31) the decomposition of the stress $\boldsymbol{\sigma}$ into its deviatoric ($\mathbb{d}$) and spherical ($\mathbb{s}$) parts, the spherical part can be factorised. The operation reveals that the equivalent stress σ_{eq}^D (31) is strictly identical to the equivalent stress σ_{eq}^{DH} (28), which was unnoticed so far:

$$\sigma_{eq}^D = \sigma_{eq}^{HD} \quad (32)$$

It is then obvious that the equivalent stress σ_{eq}^D suffers the same limitations as the equivalent stress σ_{eq}^{HD} .

As already mentioned by Yamazaki et al. (2018), it can also be noticed that:

$$\lim_{K \rightarrow 0} \sigma_{eq}^{HD} = \sigma_{eq}^{DH} \quad (33)$$

D.2 Equivalence in coercive field

Rekik et al. (2014a) developed an equivalence in coercive field: two stress states are considered equivalent if they are associated to the same coercive field for the material. Although it was unnoticed in the article, it appears that the approach leads exactly to the same definition for the equivalent stress σ_{eq}^{Re} as in the equivalence in magnetisation (28) and (31):

$$\sigma_{eq}^{Re} = \sigma_{eq}^D = \sigma_{eq}^{HD} \quad (34)$$

D.3 Equivalence in magnetostriction

Using an analytical expression for the magnetostriction strain (Daniel, 2018), Le Soudeur and Hubert (2025) proposed an equivalent stress σ_{eq}^{LS} based on an equivalence in longitudinal magnetostriction strain. They noted that the result is identical to (28):

$$\sigma_{eq}^{LS} = \sigma_{eq}^{Re} = \sigma_{eq}^D = \sigma_{eq}^{HD} \quad (35)$$

D.4 Equivalent strain approach

Some numerical formulations prefer the use of the strain tensor ε as an alternative for the stress tensor σ as the input state variable in the magneto-elastic constitutive equations. Similar equivalence approaches can be adopted. An equivalence in macroscopic energy was followed in Daniel (2017) to define the following equivalent strain ε_{eq} for magneto-elastic behaviour:

$$\varepsilon_{eq} = \frac{3}{2} \varepsilon_{//} - \frac{1}{2} \text{tr}(\varepsilon) \quad (36)$$

with $\varepsilon_{//}$ the projection of the strain tensor ε along the magnetic field direction $\mathbf{h}$ ($\varepsilon_{//} = \mathbf{h} \cdot \varepsilon \cdot \mathbf{h}$). Similarly to deviatoric equivalent stress approach (26), this definition does not require any material parameter. It can be put in a form very similar to (26):

$$\varepsilon_{eq} = \frac{3}{2} \mathbf{h} \cdot \varepsilon^d \cdot \mathbf{h} \quad (37)$$

where ε^d is the deviatoric part of the strain tensor ε .

IV. PRACTICAL IMPLEMENTATION OF AN EQUIVALENT STRESS APPROACH

Despite its simplicity of implementation, there are only few examples of implementation of an equivalent stress approach in the literature. Krebs and Daniel (2012) used it for the design of a field weakening method based on giant magnetostrictive materials for a permanent-magnet synchronous motor. It was also implemented for the determination of the effect of stress on permanent magnet synchronous motor performance (Yamazaki and Kato, 2014; Yamazaki and Aoki, 2016; Yamazaki et al., 2018, 2020). Helbling et al. (2022) used it for the interpretation of the effect of perpendicular compaction on the behaviour of electrical steel sheets for motor applications. Another application was proposed by Gueye et al. (2016) to elucidate the ferromagnetic resonance in thin films submitted to multiaxial stress state. Shimizu et al. (2023) implemented the PSD method in the case of a non-oriented Iron-Silicon sheet subjected to uniaxial compression with in-plane magnetic field.

The objective of this section is to illustrate the practical implementation of an equivalent stress approach on an example. The objective is to predict the influence of a multiaxial stress state σ on a property of interest $p(\sigma, \mathbf{H})$ knowing only this property p under uniaxial loading configurations.

A. GENERAL METHODOLOGY

The first step consists in identifying the property p of the studied material under 1D magnetoelastic configurations (uniaxial stress $\sigma_u = \sigma_u \mathbf{u} \otimes \mathbf{u}$ and magnetic field $\mathbf{H} = H \mathbf{h}$ with $\mathbf{u} = \mathbf{h}$). These 1D references results will generally be obtained from experimental characterisation. There are many examples in the literature of experimental measurements of magnetic permeability

or magnetic losses under uniaxial stress (see for instance LoBue et al. (2000); Singh et al. (2016); Karthaus et al. (2019); Domenjoud and Daniel (2023); Ouazib et al. (2024) among others). These are the only data required, in addition to the material parameters K and r for the HD (28) and Ra (27) approaches, respectively.

Let assume that the magnetoelastic problem of interest is the study of an electrical machine as illustrated in Fig.1. The property of interest p could then be the magnetic permeability. The approach is then as follows. A first distribution of stress $\sigma(P)$ and magnetic field $\mathbf{H}(P)$ is defined in the structure from an uncoupled simulation. From this distribution, the equivalent stress $\sigma_{eq}(P)$ can be calculated at each point of the structure. The property $p(\sigma, \mathbf{H})$ is then evaluated, at each point of the structure, as $p(\sigma, \mathbf{H}) = p(\sigma_{eq}, H)$, where $p(\sigma_{eq}, H)$ is interpolated from the reference 1D measurements. The property p can then be updated in the simulation and the operation is repeated until convergence.

B. ILLUSTRATION ON A FEW SIMPLE CONFIGURATIONS

The objective of this subsection is to illustrate the equivalent stress approach on a few typical configurations. For these illustrative examples, the considered property p is the hysteresis losses of a magnetic material. For the sake of simplicity, we will assume that we are only interested in the value of hysteresis losses at a frequency $f = 10$ Hz for a peak induction $B_{max} = 1$ T. Of course, in a real situation, a set of experimental measurements at various frequencies and various peak inductions would be required. The reference values of p are taken from the experimental measurements of Ouazib et al. (2024), performed on a low-carbon steel under uniaxial stress. They are shown in Fig.3. The corresponding material parameters, required for the calculation of σ_{eq}^{HD} and σ_{eq}^{Ra} are known on this material (da Silva et al., 2022). They are given in Table I.

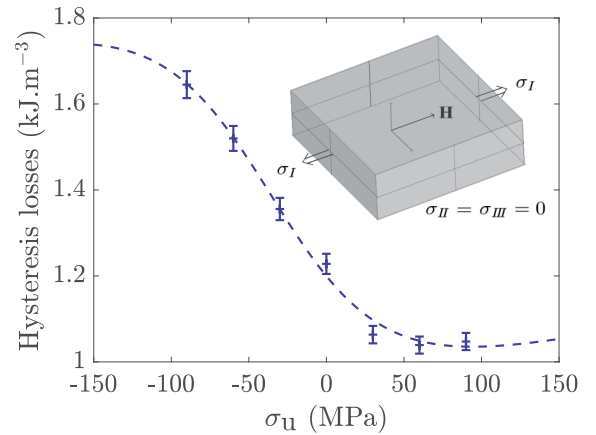


Fig. 3. Uniaxial material data to perform the equivalent stress approach: experimental measurement of hysteresis losses (sinusoidal induction at $f = 1$ Hz with peak induction value $B_{max} = 1$ T) on a low carbon steel under uniaxial stress (Ouazib et al., 2024). The inset illustrates the considered magneto-elastic configuration.

From these parameters, the value of K (29) can be determined as $K = 5.5 \cdot 10^{-8} \text{ Pa}^{-1}$.

From the 1D measurements of Fig.3 only, the property p under multiaxial loadings can be predicted following the different approaches presented in the previous section.

TABLE I. MATERIAL PROPERTIES REQUIRED FOR HD AND RA APPROACHES: SATURATION MAGNETISATION M_s , MAXIMUM LONGITUDINAL MAGNETOSTRICTION STRAIN λ_m , INITIAL ANHYSTERETIC SUSCEPTIBILITY UNDER NO APPLIED STRESS χ_0 AND UNIAXIAL STRESS AMPLITUDE FOR MAXIMUM MAGNETIC PERMEABILITY r

Parameter	M_s	λ_m	χ_0	r
Value	$1.4 \cdot 10^6$	6	5000	80
Unit	A.m ⁻¹	10 ⁻⁶	-	MPa
Model	HD			Ra
Source	da Silva et al. (2022)			

B.1 The canonical 2D-case

The first multiaxial configuration, studied in the early works of Kashiwaya (1991); Schneider and Richardson (1982); Sablik et al. (1994); Pearson et al. (2000) is the case of a biaxial stress with the magnetic field applied along one principal stress direction (Fig.2). The corresponding stress tensor can be written as $\sigma = \sigma_I \mathbf{x} \otimes \mathbf{x} + \sigma_{II} \mathbf{y} \otimes \mathbf{y}$ and $\mathbf{H} = H \mathbf{x}$.

In that case, the expressions of the various equivalent stress can be simplified. σ_{eq}^{DH} reduces to (38), σ_{eq}^{Ra} is equal to σ_{eq}^{DH} , since there is no xy shear term, σ_{eq}^{HD} reduces to (39).

$$\sigma_{eq}^{DH} = \sigma_I - \frac{1}{2} \sigma_{II} \quad (38)$$

$$\sigma_{eq}^{HD} = \sigma_I + \frac{1}{K} \ln \left[\frac{2}{1 + \exp(K \sigma_{II})} \right] \quad (39)$$

The property of interest $p(\sigma)$ is then defined as $p(\sigma_u = \sigma_{eq})$, which can be extracted from the data on Fig.3. If the PSD method is applied, $p(\sigma)$ is defined as $p(\sigma_u = \sigma_I)$, and finally, if the PDS method is applied, $p(\sigma)$ is defined as $p(\sigma_u = \frac{3}{2} d_I)$.

Fig.4, Fig.5 and Fig.6 show the predicted property p under this biaxial loading for the DH, HD and PSD approaches, respectively.

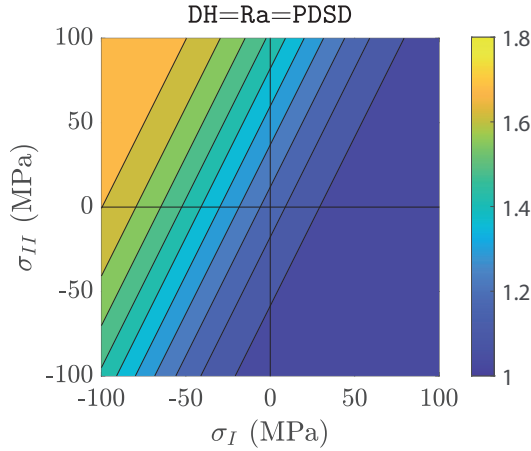


Fig. 4. Predicted hysteresis losses based on the data from Fig.3, and the use of the equivalent stress σ_{eq}^{DH} for a biaxial stress with the magnetic field aligned along the first principal stress direction (illustration on Fig.2). In that configuration, DH, Ra and PDS methods are equivalent.

The isovalue of the hysteresis losses for the DH approach are parallel lines, resembling the experimental measurements shown in Aydin et al. (2019b,a). The HD approach shows curved lines, where the effect of a compressive stress in the direction perpendicular to the magnetic field has no effect on the response of the material. In contrast, a tensile stress in the direction perpendicular to the magnetic field has a significant effect on the response of the material. Such evolutions were observed for the magnetic susceptibility under biaxial stress (Le Soudeur and Hubert,

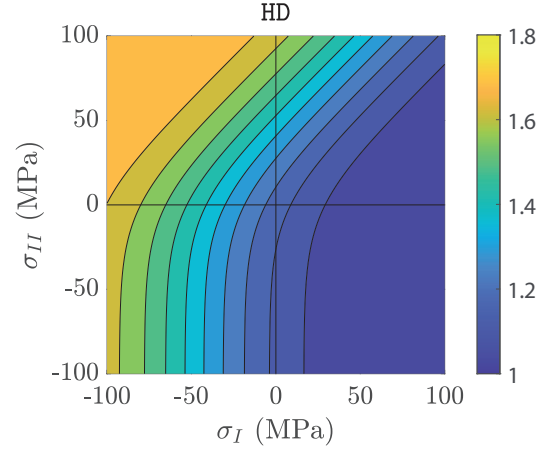


Fig. 5. Predicted hysteresis losses based on the data from Fig.3, and the use of the equivalent stress σ_{eq}^{HD} for a biaxial stress with the magnetic field aligned along the first principal stress direction (illustration on Fig.2). For the implementation of the approach, $\mathbf{t}_2$ was taken as the direction $\mathbf{z}$.

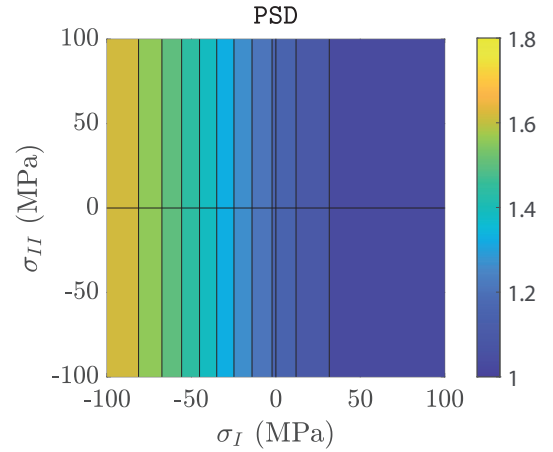


Fig. 6. Predicted hysteresis losses based on the data from Fig.3, and the use of the principal stress decomposition (PSD) method for a biaxial stress with the magnetic field aligned along the first principal stress direction (illustration on Fig.2).

2025). The PSD approach shows absolutely no sensitivity to the second principal stress, since the field is aligned with the first principal stress. This results in vertical isovalues in Fig.6.

B.2 A uniaxial-stress multiaxial configuration

Another multiaxial configuration is the case of a uniaxial stress $\sigma_u = \sigma_u \mathbf{x} \otimes \mathbf{x}$ combined with a magnetic field $\mathbf{H} = H \mathbf{h}$ with $\mathbf{h} = {}^t[\cos \theta \sin \theta 0]$. The expressions of the various equivalent stress can be simplified. σ_{eq}^{DH} , σ_{eq}^{Ra} and σ_{eq}^{HD} reduce to (40), (41) and (42), respectively. In the case of the HD approach, the choice of $\mathbf{t}_1$ and $\mathbf{t}_2$ is not unique, leading to different expressions for σ_{eq}^{HD} and hence for p . In (42), $\mathbf{t}_2$ was chosen as $\mathbf{z}$.

$$\sigma_{eq}^{DH} = \frac{3}{2} \sigma_u \left(\cos^2 \theta - \frac{1}{3} \right) \quad (40)$$

$$\sigma_{eq}^{Ra} = \begin{cases} r - \sqrt{r^2 + r \sigma_u (1 - 3 \cos^2 \theta) + \frac{\sigma_u^2}{4} (1 + 3 \cos^2 \theta)}, & \text{if } \sigma_{eq}^{DH} \leq r \\ r + \sqrt{r^2 + r \sigma_u (1 - 3 \cos^2 \theta) + \frac{\sigma_u^2}{4} (1 + 3 \cos^2 \theta)}, & \text{otherwise} \end{cases} \quad (41)$$

$$\sigma_{eq}^{HD} = \sigma_u \cos^2 \theta + \frac{1}{K} \ln \left[\frac{2}{1 + \exp(K \sigma_u \sin^2 \theta)} \right] \quad (42)$$

The property of interest $p(\sigma)$ is then defined as $p(\sigma_u = \sigma_{eq})$, interpolated from Fig.3. If the PSD method is applied, $p(\sigma)$ is defined as $p(\sigma_u = \sigma_I \cos \theta + \sigma_{II} \sin \theta)$, and finally, the PDS method is applied, $p(\sigma)$ is defined as $p(\sigma_u = d_I \cos \theta + d_{II} \sin \theta)$. The predicted property p according to the various equivalent stress approaches, for a tensile stress $\sigma_u = 80$ MPa, is shown in Fig.7

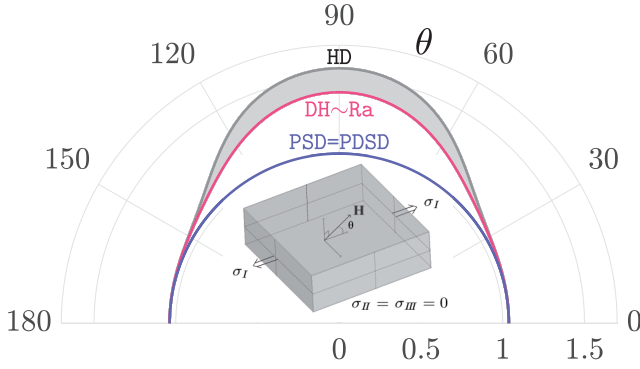


Fig. 7. Predicted hysteresis losses based on the data from Fig.3, and the use of equivalent stress methods for a uniaxial tensile stress $\sigma_u = 80$ MPa with the magnetic field oriented at an angle θ from the stress direction. The results are plotted in polar coordinates. The inset illustrates the considered magneto-elastic configuration.

PSD and PDS methods are equivalent for this configuration. They show no sensitivity to the angle of application of the electric field, resulting in a circle on Fig.7. Such an insensitivity is unexpected under such loading conditions (see for instance Kai et al. (2011)). DH and Ra approaches, while their expressions differ, provide very similar predictions in that case. The HD method does not provide a unique definition of σ_{eq}^{HD} , depending on the choice for the directions $\mathbf{t}_1$ and $\mathbf{t}_2$ in (28). The range of predictions provided by the HD approach are shown as a grey area, obtained when the two perpendicular unit vectors cover all the possible definitions for $\mathbf{t}_1$ and $\mathbf{t}_2$. Choosing $\mathbf{t}_2 = \mathbf{z}$ as in (42) leads to one extremum prediction, shown in darker grey in Fig.7. It is remarkable that DH provides the other extremum of the HD prediction.

B.3 Other simple configurations

Other configurations can be interesting to evaluate the features of the different equivalent stress approaches.

Hydrostatic stress. Hydrostatic stress is a purely spherical stress defined as $\sigma = \sigma_o \mathbf{I}$. It is remarkable that the three equivalent stress approaches presented in this paper, together with the PDS are insensitive to hydrostatic stress:

$$\sigma_{eq}^{DH}(\sigma_o \mathbf{I}) = \sigma_{eq}^{Ra}(\sigma_o \mathbf{I}) = \sigma_{eq}^{HD}(\sigma_o \mathbf{I}) = 0 \quad (43)$$

This is expected considering that the magnetostriction is isochoric, and it is in agreement with experimental observations, as long as volume magnetostriction is not involved (Bozorth, 1951). In contrast, PSD implies a significant evolution of p with hydrostatic pressure, which constitutes a strong warning to its practical use (see Fig.8).

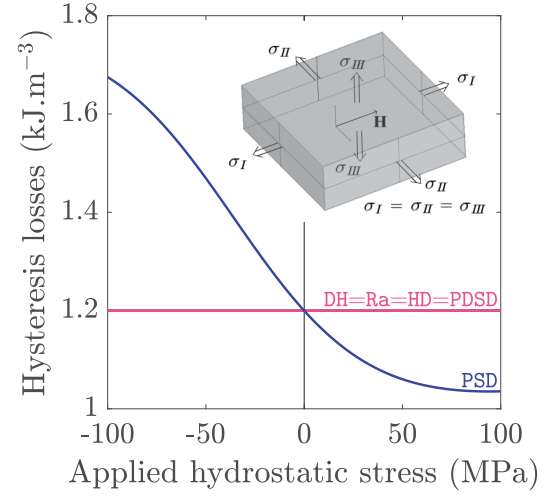


Fig. 8. Predicted hysteresis losses based on the data from Fig.3, and the use of equivalent stress methods for a hydrostatic stress. The inset illustrates the considered magneto-elastic configuration. All approaches predict no effect of hydrostatic stress except the PSD method. The inset illustrates the considered magneto-elastic configuration.

Equi-biaxial configuration. The equibiaxial stress is defined as $\sigma = \sigma_o(\mathbf{x} \otimes \mathbf{x} + \mathbf{y} \otimes \mathbf{y})$ and the magnetic field is taken parallel to $\mathbf{x}$. This case is a particular case of the canonical-2D case, along the diagonal $\sigma_I = \sigma_{II}$. The predicted response is shown in Fig.9. As already mentioned in Fig.4, DH, Ra and PDS methods are equivalent in that case. The range of predictions for the HD approach is shown as a grey area. It is again observed that the choice $\mathbf{t}_2 = \mathbf{z}$ provides one extremum of the solutions, while DH provides the other extremum of the HD prediction.

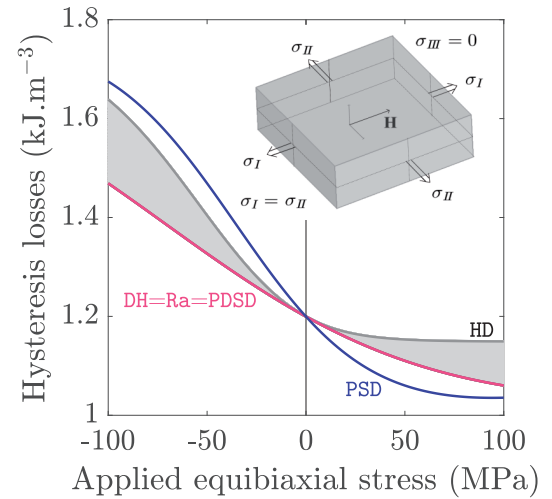


Fig. 9. Predicted hysteresis losses based on the data from Fig.3, and the use of equivalent stress methods for an equibiaxial stress with the magnetic field along a principal stress direction. The grey area illustrates the range of possible results for the HD approach due to the non-uniqueness for the definition of the perpendicular directions. The inset illustrates the considered magneto-elastic configuration.

Pure shear loading. Two pure shear loading conditions are considered. In the first one (shear I), the shear stress is applied at 45° with respect to the magnetic field direction $\mathbf{h}$: $\sigma = \sigma_o(\mathbf{x} \otimes \mathbf{x} - \mathbf{y} \otimes \mathbf{y})$. This is another particular case of the canonical-2D case, along the second diagonal $\sigma_I = -\sigma_{II}$. The corresponding predicted response is shown in Fig.10. DH, Ra and PDS methods are again equivalent in that case. The range of predictions for the HD approach is shown as a grey area. It is again observed that the choice $\mathbf{t}_2 = \mathbf{z}$ provides one extremum of the solutions, while DH provides the other extremum of the HD prediction.

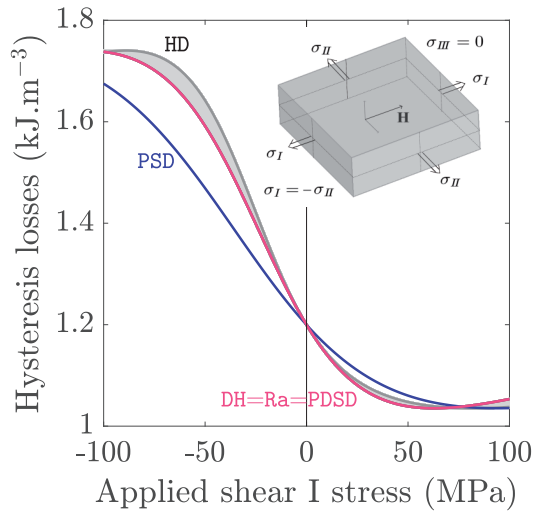


Fig. 10. Predicted hysteresis losses based on the data from Fig.3, and the use of equivalent stress methods for a pure shear stress with the magnetic field at 45° from the shear component. The grey area illustrates the range of possible results for the HD approach due to the non-uniqueness for the definition of the perpendicular directions. The inset illustrates the considered magneto-elastic configuration.

In a second configuration (shear II), the shear stress is applied along the magnetic field direction $\mathbf{h}$: $\boldsymbol{\sigma} = \sigma_o(\mathbf{x} \otimes \mathbf{y} + \mathbf{y} \otimes \mathbf{x})$. The corresponding predicted response is shown in Fig.11.

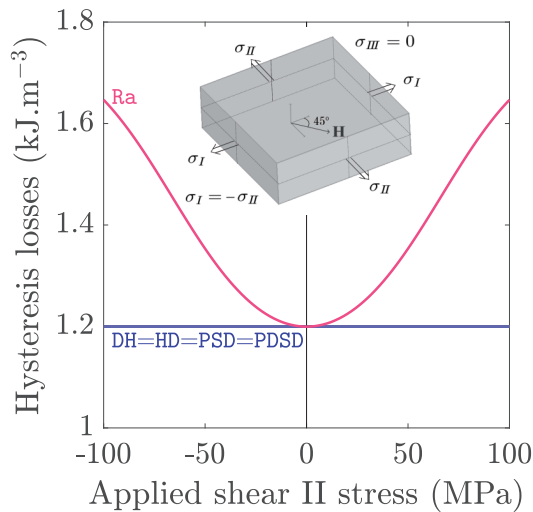


Fig. 11. Predicted hysteresis losses based on the data from Fig.3, and the use of equivalent stress methods for a pure shear stress with the magnetic field along the shear component. The Ra approach is the only approach able to capture an effect of shear stress in that configuration. The inset illustrates the considered magneto-elastic configuration.

All approaches exhibit no sensitivity to this shear II configuration except the Ra approach. The trend predicted by the Ra approach resembles the experimental observations in Aydin et al. (2019b).

V. SUMMARY AND CONCLUSION

This article reviewed the available approaches to define equivalent stress approach for the magneto-elastic behaviour of ferro-magnetic materials. While these approaches rely on strong approximations, they are powerful tools to evaluate the response of materials to multiaxial loadings, based on limited knowledge obtained from uniaxial experiments only. It was shown that some of the equivalent stresses proposed in the literature turn out to be identical.

The main features of the different approaches are summarised

in Table II. The historical approaches SR, K, SR have been seminal in the development of the concept of magneto-elastic equivalent stress approaches, but they are limited to very specific two-dimensional configurations. They cannot therefore be regarded as general multiaxial equivalent stress criteria. The PSD approach suffers from the major drawback of being sensitive to hydrostatic pressure, which can lead to significant prediction errors. Four main approaches emerge: the deviatoric equivalent stress (DH), the extended deviatoric equivalent stress (Ra), the magnetisation deviatoric equivalent stress HD, and the principal deviatoric stress decomposition (PDSD) method. All four are fully multiaxial in nature. An important feature is that the corresponding equivalent stresses reduce consistently to the input reference data in the case of a uniaxial stress applied along the field direction, which is the reference configuration. The HD approach is not uniquely defined for a given magneto-mechanical configuration, which requires special care in its implementation. Finally, the Ra approach is the only one to date able to consider the effects of a shear stress component along the magnetic field direction.

So far, these approaches do not incorporate the anisotropy of material behaviour. In analogy with the development of the anisotropic Hill criterion following the isotropic Von Mises criterion, there is undoubtedly room for the formulation of anisotropic magneto-elastic equivalent stresses. Of course, multiaxial characterisation setups are instrumental for validating these methods and differentiating their specific features. The development of such setups under complex loading conditions should therefore be strongly encouraged. Finally, it should be stressed that equivalent stress approaches provide efficient tools for a fast evaluation of multiaxial effects, but they cannot replace fully coupled multiaxial constitutive models.

ACKNOWLEDGEMENT

This article is dedicated to the memory of Dr. Martin Sablik, as a tribute to his inspirational work on magneto-mechanical equivalent stresses, and in remembrance of our fruitful discussion on the subject. I would like to thank all the colleagues who contributed to the development of the equivalent stress approaches summarised in this article. Special thanks go to Robin Adam for his help in sketching the 2D configurations.

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TABLE II. SUMMARY OF THE DIFFERENT EQUIVALENT STRESS APPROACHES

Equivalent stress designation and original reference	Definition (eq. nb)	Full multiaxiality (y/n)	Uniqueness (y/n)	Material dependent (y/n)	Sensitive to hydrostatic stress (y/n)	Sensitive to shear along $\mathbf{h}$ (y/n)
SR (Schneider and Richardson, 1982)	(14)	n	y	n	n.a.	n.a.
K (Kashiwaya, 1991)	(15)	n	y	y	n.a.	n.a.
S (Sablík et al., 1994)	(16)	n	y	n	n.a.	n.a.
PSD (Nakano et al., 2009)	(17)	y	y	n	y	n
PDSD	(18)	y	y	n	n	n
DH (Daniel and Hubert, 2009)	(26)	y	y	n	n	n
HD (Hubert and Daniel, 2011)	(28) and (31)	y	n	y	n	n
Ra (Rasilo et al., 2019)	(27)	y	y	y	n	y

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