A Note on the Effective Magnetic Permeability of Polycrystals

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Many classical estimates for the effective behavior of heterogeneous materials can be reinterpreted in terms of inclusion problems. However, in the case of cubic polycrystals, a cubic permeability tensor for single crystals has to be written. In the framework of linear behavior, the description of the cubic symmetry reduces to isotropy. The heterogeneity of polycrystals, which results from single crystal anisotropy, cannot be described, and the classical estimates for the overall behavior of heterogeneous materials cannot be used. In this paper, we propose a particular description of the cubic symmetry for the magnetic permeability. We then derive estimates for the effective permeability of polycrystals from the solution of the basic inclusion problem, for both macroscopically isotropic and anisotropic polycrystals.

Index Terms—**Cubic symmetry, effective properties, homogenization, inclusion, magnetic permeability, polycrystal.**

I. Introduction

THE determination of the effective properties of heterogeneous materials is a long-standing problem in many fields of physics. The purpose is to deduce the material constants of a fictive homogeneous material equivalent to the heterogeneous real one from the properties of its constituents and some assumptions on the microstructure. In the case of magnetic properties, the problem can be presented in the following way: the material is constituted of n phases i for which the behavior is known. Under the assumption of linear magnetic properties, the constitutive law is written:

$$\mathbf{B^i} = \underline{\underline{\mu}}^{\mathbf{i}}\mathbf{H^i} \tag{1}$$

where $\mathbf{B^i}$ and $\mathbf{H^i}$ are the magnetic induction and the magnetic field in the phase i and $\underline{\underline{\mu}}^{\mathbf{i}}$ its magnetic permeability.[1] The objective is to define the effective permeability tensor $\underline{\underline{\mu}}^{\mathbf{eff}}$ of the heterogeneous material, linking the mean magnetic induction in the material $\overline{\mathbf{B}}$ to the mean magnetic field in the material $\overline{\mathbf{H}}$, according to (2)[2]:

$$\overline{\mathbf{B}} = \langle \mathbf{B^i} \rangle = \underline{\underline{\mu}}^{\mathbf{eff}}\overline{\mathbf{H}} = \underline{\underline{\mu}}^{\mathbf{eff}}\langle \mathbf{H^i} \rangle. \tag{2}$$

The simplest general estimates, the Wiener bounds [1], are obtained assuming uniform magnetic field [$\mathbf{H^i} = \overline{\mathbf{H}}$, see (3)] or magnetic induction [$\mathbf{B^i} = \overline{\mathbf{B}}$, see (4)] within the material

$$\overline{\mathbf{B}} = \langle \mathbf{B^i} \rangle = \langle \underline{\underline{\mu}}^{\mathbf{i}}\mathbf{H^i} \rangle = \langle \underline{\underline{\mu}}^{\mathbf{i}} \rangle \overline{\mathbf{H}} \tag{3}$$

$$\overline{\mathbf{H}} = \langle \mathbf{H^i} \rangle = \left\langle \underline{\underline{\mu}}^{\mathbf{i}^{-1}}\mathbf{B^i} \right\rangle = \left\langle \underline{\underline{\mu}}^{\mathbf{i}^{-1}} \right\rangle \overline{\mathbf{B}}. \tag{4}$$

The Wiener lower and upper bounds are thus given by relation (5):

$$\underline{\underline{\mu}}^{\mathbf{eff}}_{W-} = \left\langle \underline{\underline{\mu}}^{\mathbf{i}^{-1}} \right\rangle^{-1} \quad \text{and} \quad \underline{\underline{\mu}}^{\mathbf{eff}}_{W+} = \langle \underline{\underline{\mu}}^{\mathbf{i}} \rangle. \tag{5}$$

An alternative estimate can be obtained assuming uniform magnetization within the material[3] ($\mathbf{M^i} = \overline{\mathbf{M}}$):

$$\overline{\mathbf{H}} = \langle \mathbf{H^i} \rangle = \left\langle \underline{\underline{\chi}}^{\mathbf{i}^{-1}}\mathbf{M^i} \right\rangle = \left\langle \underline{\underline{\chi}}^{\mathbf{i}^{-1}} \right\rangle \overline{\mathbf{M}} \tag{6}$$

leading to the following estimate for the effective permeability tensor:

$$\underline{\underline{\mu}}^{\mathbf{eff}}_{UM} = \mu_0 \underline{\underline{I}} + \langle (\underline{\underline{\mu}}^{\mathbf{i}} - \mu_0 \underline{\underline{I}})^{-1} \rangle^{-1}. \tag{7}$$

Hashin and Shtrikman [2] derived more restrictive bounds in the case of isotropic biphasic composites. Another estimate, based on a geometric averaging operation, has been proposed by Lichtenecker [3]. Other estimates, based on more complicated hypotheses on the microstructure, have been proposed [4]–[8]. These bounds or estimates have been mainly developed for composite materials, with a limited number of distinct phases, mostly biphasic materials. The problem of polycrystalline media is rarely treated.[4] Some authors addressed the case of polycrystalline media [9]–[12], but the crystalline symmetry is limited to transverse isotropy in these contributions. The cubic symmetry is not taken into account. A way to describe the cubic crystalline symmetry for the magnetic behavior is proposed in this paper (Section IV). Based on the classical solution for the inclusion problem (Section II), several estimates for the effective behavior of heteregeneous materials can be found. The classical bounds and estimates for biphasic materials are recovered (Section III) and estimates for polycrystals with cubic crystalline symmetry are derived (Section V).

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[1] In the anisotropic case, the magnetic permeability is a second order tensor.

[2] The operator $\langle \cdot \rangle$ denotes an average operation over the volume.

[3] $\mathbf{M^i}$ and $\overline{\mathbf{M}}$ are the magnetization vectors respectively in phase i and in the material. $\underline{\underline{\chi}}^{\mathbf{i}}$ denotes the second order susceptibility tensor of phase i, verifying: $\mu_0 \underline{\underline{\chi}}^{\mathbf{i}} = \underline{\underline{\mu}}^{\mathbf{i}} - \mu_0 \underline{\underline{I}}$, $\underline{\underline{I}}$ denotes the second order identity tensor, and μ_0 the vacuum permeability ($\mu_0 = 4\pi\ 10^{-7}$ H/m).

[4] In the case of anisotropic single crystal behavior, the polycrystal can be seen as a n-phasic material. Each phase differs from another only by its crystallographic orientation.

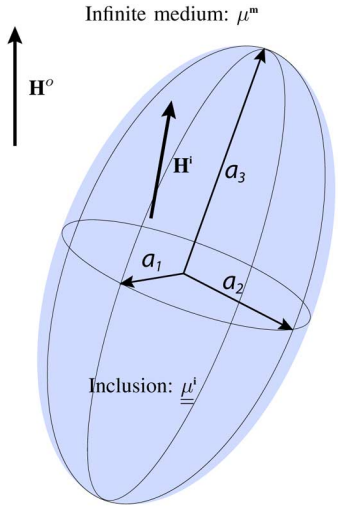

Fig. 1. Basic inclusion problem.

II. Basic Inclusion Problem

Many homogenization models are based on the resolution of the problem of an ellipsoidal inclusion embedded in an infinite medium. The solution of this problem is briefly reminded hereafter.

The infinite medium, as well as the inclusion, is supposed to be homogeneous. The magnetic behavior is supposed linear and, in the case of the infinite medium, isotropic (permeability $\mu^{\mathbf{m}}$ and $\underline{\underline{\mu}}^{\mathbf{i}}$). In such conditions, the magnetic field $\mathbf{H}^{\mathbf{i}}$ in the inclusion is homogeneous and can be derived as a function of the applied field $\mathbf{H}^o$ (8)

$$\mathbf{H}^{\mathbf{i}} = \mu^{\mathbf{m}} \left(\mu^{\mathbf{m}}(\underline{\underline{I}} - \underline{\underline{N}}) + \underline{\underline{N}}\underline{\underline{\mu}}^{\mathbf{i}}\right)^{-1} \mathbf{H}^o. \tag{8}$$

The tensor $\underline{\underline{N}}$ is diagonal and called the demagnetizing tensor. The demagnetizing factors $N_j\,(j = \{1,2,3\})$ are computed from the following elliptic integral [13]:

$$N_j = \frac{a_1 a_2 a_3}{2} \int_0^\infty \frac{ds}{\left(s + a_j^2\right)\sqrt{\left(s + a_1^2\right)\left(s + a_2^2\right)\left(s + a_3^2\right)}} \tag{9}$$

where a_1, a_2, and a_3 denote the ellipsoid's semi-axes (see Fig. 1). One can notice $N_1 + N_2 + N_3 = 1$. In the general case, numerical computations are needed to obtain the demagnetizing factors. In the case of a spherical inclusion, the expression of the demagnetizing factors can be derived analytically

$$\underline{\underline{N}} = \frac{1}{3}\underline{\underline{I}}. \tag{10}$$

For the sake of simplicity, we will focus our study on that case. The magnetic field $\mathbf{H}^{\mathbf{i}}$ in the inclusion can then be written:

$$\mathbf{H}^{\mathbf{i}} = 3\mu^{\mathbf{m}}\left(2\mu^{\mathbf{m}}\underline{\underline{I}} + \underline{\underline{\mu}}^{\mathbf{i}}\right)^{-1}\mathbf{H}^o. \tag{11}$$

The effective behavior of an heterogeneous medium can be defined through the definition of its magnetic induction $\overline{\mathbf{B}}$, considering that each phase behaves as an inclusion[5] embedded in an infinite medium with permeability $\mu^{\mathbf{m}}$ (12)

$$\begin{aligned}\overline{\mathbf{B}} &= \underline{\underline{\mu}}^{\mathbf{eff}}\langle\mathbf{H}^{\mathbf{i}}\rangle \\ &= \langle\underline{\underline{\mu}}^{\mathbf{i}}\mathbf{H}^{\mathbf{i}}\rangle \\ &= \left\langle 3\mu^{\mathbf{m}}\underline{\underline{\mu}}^{\mathbf{i}}\left(2\mu^{\mathbf{m}}\underline{\underline{I}} + \underline{\underline{\mu}}^{\mathbf{i}}\right)^{-1}\mathbf{H}^o\right\rangle \\ &= \left\langle 3\mu^{\mathbf{m}}\underline{\underline{\mu}}^{\mathbf{i}}\left(2\mu^{\mathbf{m}}\underline{\underline{I}} + \underline{\underline{\mu}}^{\mathbf{i}}\right)^{-1}\right\rangle\mathbf{H}^o.\end{aligned} \tag{12}$$

The applied field in the inclusion problem $\mathbf{H}^o$ (different from the mean magnetic field $\overline{\mathbf{H}}$ in the real heterogeneous material) is defined with respect to (13):

$$\mathbf{H}^o \quad / \quad \langle\mathbf{H}^{\mathbf{i}}\rangle = \left\langle 3\mu^{\mathbf{m}}\left(2\mu^{\mathbf{m}}\underline{\underline{I}} + \underline{\underline{\mu}}^{\mathbf{i}}\right)^{-1}\right\rangle\mathbf{H}^o = \overline{\mathbf{H}}. \tag{13}$$

$\mathbf{H}^o$ is then given by (14):

$$\mathbf{H}^o = \left\langle 3\mu^{\mathbf{m}}\left(2\mu^{\mathbf{m}}\underline{\underline{I}} + \underline{\underline{\mu}}^{\mathbf{i}}\right)^{-1}\right\rangle^{-1}\overline{\mathbf{H}}. \tag{14}$$

Using (12) and (14), the effective permeability is defined according to (15):

$$\underline{\underline{\mu}}^{\mathbf{eff}} = \left\langle\underline{\underline{\mu}}^{\mathbf{i}}(2\mu^{\mathbf{m}}\underline{\underline{I}} + \underline{\underline{\mu}}^{\mathbf{i}})^{-1}\right\rangle \cdot \left\langle\left(2\mu^{\mathbf{m}}\underline{\underline{I}} + \underline{\underline{\mu}}^{\mathbf{i}}\right)^{-1}\right\rangle^{-1}. \tag{15}$$

Several estimates of the effective permeability can be obtained through different choices of the permeability $\mu^{\mathbf{m}}$ of the infinite medium. The appropriate choice will depend on the microstructure of the heterogeneous material. This approach enables to recover classical estimates in the case of biphasic composites (Section III). However, as shown in Section IV, it cannot be applied to most of polycrystals unless a definition for a cubic single crystal permeability is proposed.

III. Permeability of Biphasic Composites

In the case of a biphasic material with isotropic constituents (phase 1 with volumetric fraction f_1 together with phase 2 with volumetric fraction $f_2\,(f_1 + f_2 = 1)$), (15) can be written:

$$\mu^{\mathbf{eff}} = \frac{f_1\dfrac{\mu_1}{2\mu^{\mathbf{m}} + \mu_1} + (1 - f_1)\dfrac{\mu_2}{2\mu^{\mathbf{m}} + \mu_2}}{\dfrac{f_1}{2\mu^{\mathbf{m}} + \mu_1} + \dfrac{1 - f_1}{2\mu^{\mathbf{m}} + \mu_2}}. \tag{16}$$

Any value of $\mu^{\mathbf{m}}$ (from 0 to infinity) can be considered, representing various kinds of microstructure for the composite. It can be noticed that in the case of an homogeneous medium, for which $f_1 = 1$ (resp. $f_1 = 0$), (16) defines an effective permeability $\mu^{\mathbf{eff}} = \mu_1$ (resp. $\mu^{\mathbf{eff}} = \mu_2$), whatever the choice of $\mu^{\mathbf{m}}$.

In the general case of biphasic materials, some particular choices of $\mu^{\mathbf{m}}$ allow to recover some classical estimates.

[5]The shape chosen for the inclusion depends on the distribution of the phases in the composite. Spherical inclusions are associated to an isotropic distribution and ellipsoidal inclusions to an anisotropic one. This distribution of the phases is not related to the shape of the phases in the real material.

1) Wiener lower and upper bounds are obtained choosing respectively $\mu^{\mathbf{m}} = 0$ and $\mu^{\mathbf{m}} \to \infty$:

$$\mu^{\mathbf{eff}}{}_{W-} = \frac{1}{\dfrac{f_1}{\mu_1} + \dfrac{1-f_1}{\mu_2}} \tag{17}$$

$$\mu^{\mathbf{eff}}{}_{W+} = f_1\mu_1 + (1-f_1)\mu_2. \tag{18}$$

2) Hashin and Shtrikman lower and upper bounds are obtained choosing respectively (assuming $\mu_1 < \mu_2$) $\mu^{\mathbf{m}} = \mu_1$ and $\mu^{\mathbf{m}} = \mu_2$:

$$\mu^{\mathbf{eff}}{}_{HS-} = \frac{2f_1\mu_1 + (3-2f_1)\mu_2}{(3-f_1)\mu_1 + f_1\mu_2}\mu_1 \tag{19}$$

$$\mu^{\mathbf{eff}}{}_{HS+} = \frac{(1+2f_1)\mu_1 + 2(1-f_1)\mu_2}{(1-f_1)\mu_1 + (2+f_1)\mu_2}\mu_2. \tag{20}$$

As mentioned in [14], Maxwell-Garnett estimates are equivalent to Hashin and Shtrikman bounds.

3) Bruggeman estimate is obtained choosing $\mu^{\mathbf{m}} = \mu^{\mathbf{eff}}$. $\mu^{\mathbf{eff}}$ is then the solution of the following self-consistent equation:

$$f_1\frac{\mu_1 - \mu^{\mathbf{eff}}}{2\mu^{\mathbf{eff}} + \mu_1} + (1-f_1)\frac{\mu_2 - \mu^{\mathbf{eff}}}{2\mu^{\mathbf{eff}} + \mu_2} = 0. \tag{21}$$

According to these results, the classical estimates for the magnetic behavior of biphasic composites can be reinterpreted as inclusion based models. The application of such an approach to polycrystalline media is then expected to produce several estimates for polycrystal behavior. However, since the heterogeneity of polycrystals is related to single crystal anisotropy, this anisotropy has to be described.

IV. Cubic Permeability

Most ferromagnetic single crystals exhibit a body cubic centered (BCC) or face cubic centered (FCC) structure. For both structures, the magnetic behavior of the single crystal exhibits a cubic symmetry. Let us try to define the magnetic permeability second order tensor $\underline{\underline{\mu}}^{\mathbf{i}}$ of such a material, following (22), written in the crystallographic frame:

$$\mathbf{B^i} = \underline{\underline{\mu}}^{\mathbf{i}}\mathbf{H^i} \tag{22}$$

with

$$\underline{\underline{\mu}}^{\mathbf{i}} = \begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{pmatrix}_{CF}. \tag{23}$$

Since the induction $\mathbf{B^i}$ has to be parallel to the applied field $\mathbf{H^i}$ when the field is along a symmetry axis, and considering a field along $\langle 100\rangle$ directions, all extra-diagonal terms of $\underline{\underline{\mu}}^{\mathbf{i}}$ vanish. Moreover, according to the cubic symmetry, the behavior of all $\langle 100\rangle$ directions has to be identical. Then, it appears that $\mu_{11} = \mu_{22} = \mu_{33}$, and (22) reduces to

$$\mathbf{B^i} = \mu^{\mathbf{i}}\mathbf{H^i} \tag{24}$$

TABLE I
Pure Iron Single Crystal Permeability Along $\langle 100\rangle$, $\langle 110\rangle$, and $\langle 111\rangle$ Directions, According to Webster Experimental Measurements [15]

Magnetic field (A/m)	$\frac{\mu_{100}}{\mu_0}$	$\frac{\mu_{111}}{\mu_0}$	$\frac{\mu_{110}}{\mu_0}$	$\frac{3\mu_{111}+\mu_{100}}{4\mu_0}$
400	2750	1625	1800	1905
2000	835	525	615	605

with $\mu^{\mathbf{i}}$ a scalar value. Cubic symmetry is then reduced to isotropy. Magnetic permeability is on that point analogous to the compressibility modulous in elasticity: if the behavior is identical in three perpendicular directions, then it is isotropic. This conclusion points out a limitation of the use of constant second order tensors for the description of magnetic behavior. Indeed, many experimental observations reveal that cubic single crystals are not magnetically isotropic (see for instance [15] for iron and nickel or [16] for Terfenol-D). In such conditions, the definition of the effective magnetic behavior of polycrystals has no relevance since a polycrystal is defined as an aggregate of isotropic identical phases, meaning a homogeneous material.

Still in the framework of linear behavior, a particular expression for the permeability, in accordance with the cubic symmetry, can be suggested. This expression is inspired from the definition of the magneto-crystalline anisotropy energy of cubic crystals (see for instance [17]):

$$\mu^{\mathbf{i}}{}_{\gamma} = \mu_{100} + 3(\mu_{111} - \mu_{100})\left(\gamma_1^2\gamma_2^2 + \gamma_2^2\gamma_3^2 + \gamma_3^2\gamma_1^2\right) \tag{25}$$

where $[\gamma_1, \gamma_2, \gamma_3]$ are the direction cosines of the considered direction $\boldsymbol{\gamma}$. μ_{100} and μ_{111} are the extremal values of the permeability respectively along the $\langle 100\rangle$ and $\langle 111\rangle$ directions. Under this assumption, the permeability in $\langle 110\rangle$ is defined by relation (26):

$$\mu_{110} = \frac{1}{4}(3\mu_{111} + \mu_{100}). \tag{26}$$

The values of the permeability of the iron single crystal, following a secant definition, identified from Webster experimental results [15] are given in Table I.

The spatial representation of the permeability is shown in Fig. 2 for an iron single crystal under an applied field of 400 A/m.

A more refined description introduces the permeability μ_{110} in $\langle 110\rangle$ directions as a parameter:

$$\mu^{\mathbf{i}}{}_{\gamma} = \mu_{100} + 4(\mu_{110} - \mu_{100})\left(\gamma_1^2\gamma_2^2 + \gamma_2^2\gamma_3^2 + \gamma_3^2\gamma_1^2\right) + 9(3\mu_{111} + \mu_{100} - 4\mu_{110})\left(\gamma_1^2\gamma_2^2\gamma_3^2\right). \tag{27}$$

However, as shown in Fig. 3, the introduction of an additional parameter does not significantly modify the definition of the cubic permeability in the case of iron, since μ_{110} is close to expression (26).

Using the cubic definition of the permeability of the single crystal given by (25), estimates for the behavior of polycrystals can be derived from (15).

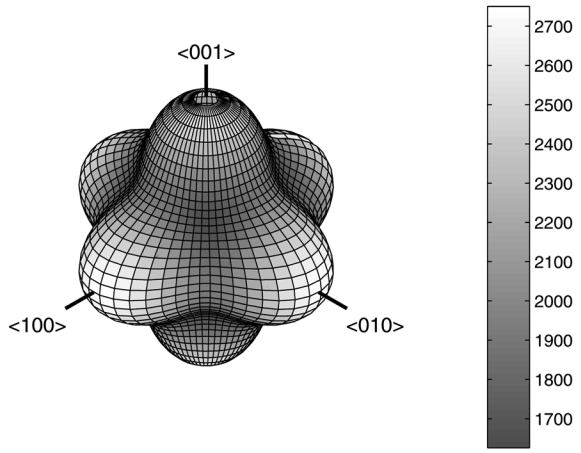

Fig. 2. Spatial representation of the relative permeability $(\mu^{\mathbf{i}}{}_{\gamma}/\mu_0)$ of an iron single crystal for $H = 400$ A/m, using a representation with two parameters.

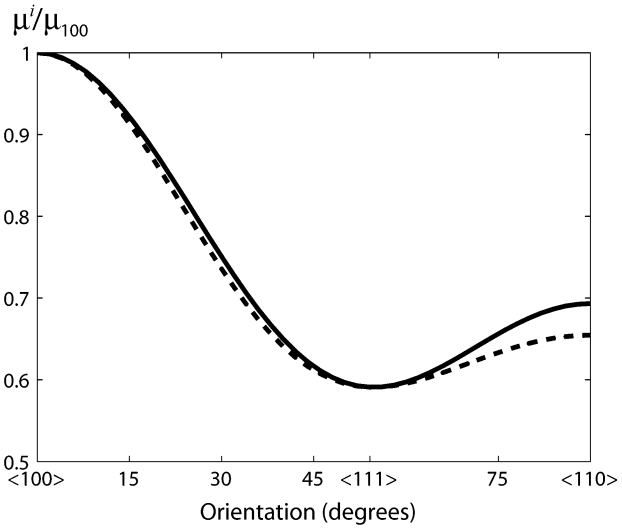

Fig. 3. Comparison between the permeability obtained using a representation with two parameters (line) and with three parameters (dashed line) for an applied field level of 400 A/m. Projection in the plane $y = z$, including directions [100], [111], and [011].

V. Permeability of Polycrystals

The effective permeability in the direction $\boldsymbol{\gamma}$ can be obtained through a projection of relation (15) along $\boldsymbol{\gamma}$:

$$\mu^{\mathbf{eff}}{}_{\gamma} = \frac{\left\langle \dfrac{\mu^{\mathbf{i}}{}_{\gamma}}{2\mu^{\mathbf{m}} + \mu^{\mathbf{i}}{}_{\gamma}} \right\rangle}{\left\langle \dfrac{1}{2\mu^{\mathbf{m}} + \mu^{\mathbf{i}}{}_{\gamma}} \right\rangle}. \tag{28}$$

If the self-consistent method is chosen, the infinite medium permeability is not a constant but the effective permeability $(\mu^{\mathbf{m}} = \mu^{\mathbf{eff}}{}_{\gamma})$, so that (15) becomes:

$$\mu^{\mathbf{eff}}{}_{\gamma} = \frac{\left\langle \dfrac{\mu^{\mathbf{i}}{}_{\gamma}}{2\mu^{\mathbf{eff}}{}_{\gamma} + \mu^{\mathbf{i}}{}_{\gamma}} \right\rangle}{\left\langle \dfrac{1}{2\mu^{\mathbf{eff}}{}_{\gamma} + \mu^{\mathbf{i}}{}_{\gamma}} \right\rangle}. \tag{29}$$

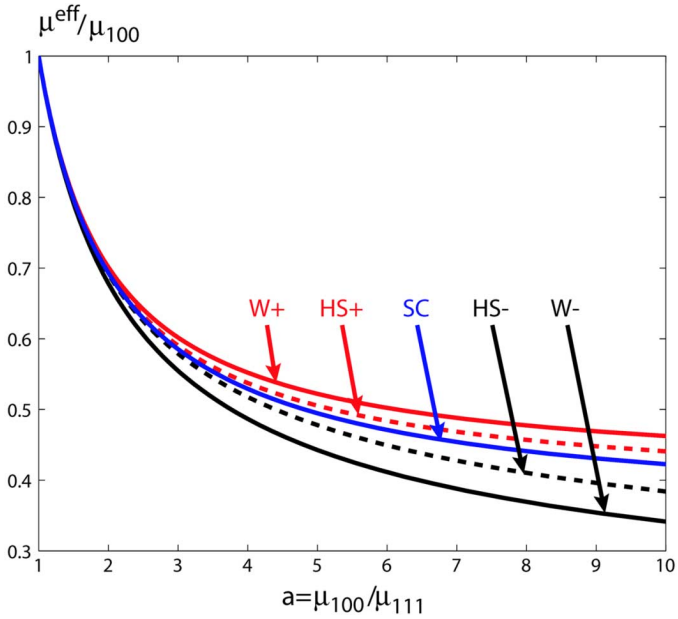

Fig. 4. Normalized effective magnetic permeability of an isotropic polycrystal as a function of the single crystal anisotropy ratio $a = \mu_{100}/\mu_{111}$: Wiener, Hashin & Shtrikman bounds and self-consistent estimate.

Equation (29) can be written in the form of the classical Bruggeman relation (30)

$$\left\langle \frac{\mu^{\mathbf{i}}{}_{\gamma} - \mu^{\mathbf{eff}}{}_{\gamma}}{2\mu^{\mathbf{eff}}{}_{\gamma} + \mu^{\mathbf{i}}{}_{\gamma}} \right\rangle = 0. \tag{30}$$

A. Isotropic Polycrystals

If the crystallographic orientation of grains in the polycrystal is random, the macroscopic behavior is isotropic. The magnetic permeability is then defined by a scalar value $\mu^{\mathbf{eff}}$. The value of $\mu^{\mathbf{eff}}$ has been estimated according to several homogenization schemes (Wiener, Hashin and Shtrikman, self-consistent) as a function of the single crystal anisotropy ratio $a(a = \mu_{100}/\mu_{111})$. The results are presented in Fig. 4. The case of the upper Wiener bound is particularly easy to calculate analytically and can be expressed as follows:

$$\mu^{\mathbf{eff}}{}_{W+} = \frac{2}{5}\mu_{100} + \frac{3}{5}\mu_{111}. \tag{31}$$

As far as low anisotropy ratio $(a < 2)$ are considered, the Wiener lower and upper bounds are very close, and all the homogenization schemes give similar results. When considering higher single crystal anisotropy ratio, the differences between the schemes become more sensitive: for $a = 5$, the difference between the Wiener lower and upper bounds is more than 15%.

B. Anisotropic Polycrystals

In most cases, the macroscopic anisotropy of polycrystals is the result of the combination of the single crystal anisotropy to crystallographic texture. The crystallographic texture is described through an orientation distribution function (ODF), representative of the orientations of grains in the polycrystal [18]. A scanning electron microscope (SEM), with an electron back

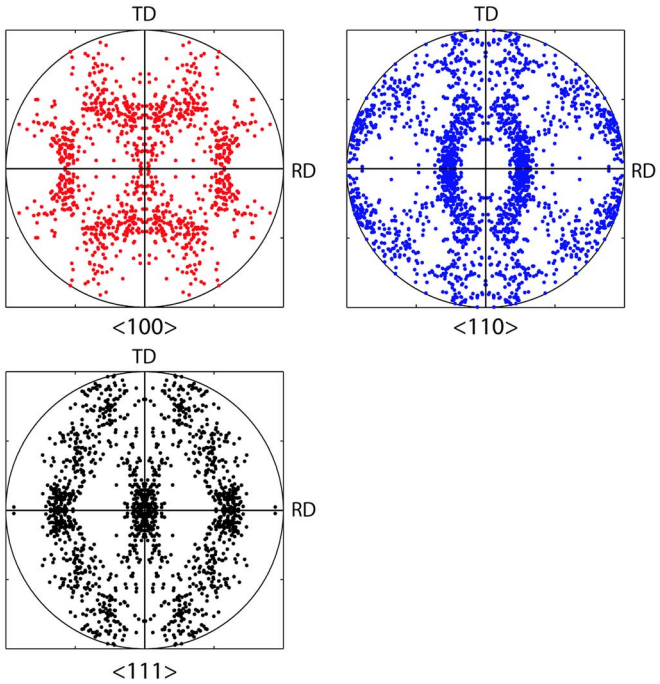

Fig. 5. $\langle 100\rangle, \langle 110\rangle$, and $\langle 111\rangle$ discrete pole figures for a nonoriented 3% silicon-iron steel.

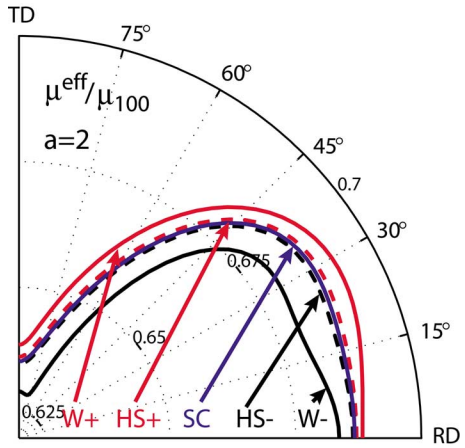

Fig. 6. Normalized magnetic permeability in the sheet plane: Wiener, Hashin & Shtrikman bounds and self-consistent estimate for an anisotropy ratio $a = 2$.

scatter diffraction (EBSD) measurement system, can provide a discrete orientation data file for the crystallographic texture of a given material. An example of pole figures, using 396 orientations, is shown for a nonoriented 3% silicon-iron steel in Fig. 5.

This discrete ODF can be used in (28) or (30) for the calculation of the effective permeability of anisotropic polycrystals. The corresponding results, according to several estimates, are shown in Figs. 6 and 7, respectively, for a single crystal anisotropy ratio $a = 2$ and $a = 5$. It must be noticed that the scales of polar Figs. 6 and 7 are not centered on the origin. This choice has been made in order to highlight the differences between the estimates.

It can be seen that the proposed approach allows the description of the macroscopic anisotropy of texturized polycrystals. As expected, for low anisotropy ratio a, corresponding to low

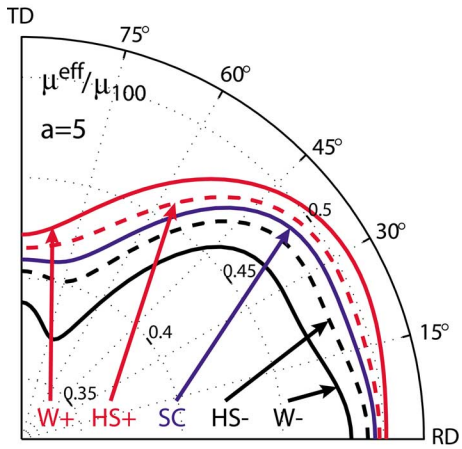

Fig. 7. Normalized magnetic permeability in the sheet plane: Wiener, Hashin & Shtrikman bounds and self-consistent estimate for an anisotropy ratio $a = 5$.

applied fields, the contrast between the several estimates is tiny. In Fig. 6, Hashin and Shtrikman bounds and self-consistent estimates are very close. But for higher anisotropy ratio, differences between the different approaches becomes significant. Moreover, the description of the macroscopic anisotropy is not strictly identical from one approach to another. Then, it becomes important to choose an appropriate homogenization model for the definition of the effective permeability.

VI. Discussion

The choice of the most appropriate model (namely the appropriate value of $\mu^{\mathbf{m}}$) for a given heterogeneous material is a key point, particularly for high anisotropy ratio of the single crystal.

Experimental results could help to evaluate the different possible choices. Unfortunately, the result is expected to be strongly dependent on the microstructure of the particular chosen material. Moreover, to our knowledge, no publication associates macroscopic polycrystalline permeability measurement to corresponding single crystals data. As an example, the experimental results from [19], for a material with a similar crystallographic texture of the one presented in Fig. 5, have been replotted[6] in Fig. 8, to highlight the anisotropic behavior.

For very low magnetic field, the relative anisotropy has a similar appearance in experiment and modeling. It corresponds to the linear behavior stage for the material. As soon as the magnetic field level reaches about 200 A/m, the nonlinear stage begins, and the hypotheses of the proposed model do not apply.

A second strategy to choose the value of $\mu^{\mathbf{m}}$ in the model would be to consider the particular microstructure of the modelled material. In the case of biphasic materials, if the microstructure is constituted of inclusions of phase 1 embedded in a continuous matrix of phase 2, the choice of the Hashin & Shtrikman bound with $\mu^{\mathbf{m}} = \mu_2$ is expected to give accurate results. In the case of polycrystals, the self-consistent scheme is known to be well suited to random grain microstructure. It is probably the method to recommend. Unfortunately, it is also the most complicated to implement since it is based on

[6]It must be noticed that the single crystal permeability has not been characterized, so that the measured permeability has been normalized to the value obtained in the rolling direction (RD).

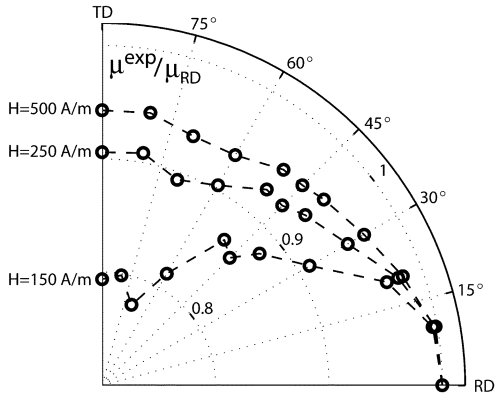

Fig. 8. Normalized magnetic permeability in the sheet plane: experimental results for a nonoriented 3% silicon-iron steel [19].

an iterative procedure. For simplicity reasons, the HS bounds can be preferred. If a rough estimate is sufficient, the formula given by (31) is particularly simple, but does not incorporate any anisotropy information.

VII. Conclusion

A model for the prediction of the effective magnetic permeability of heterogeneous linear materials has been presented. It is based on the solution of the basic inclusion problem. This approach enables to recover some classical estimates obtained under different assumptions for polyphasic materials. In the case of linear cubic polycrystals, the classical description of the permeability through a second order tensor has been shown to be inefficient. A formulation has been proposed to describe the local anisotropy of cubic single crystals. Combined with the grain orientations, this approach allows the prediction of the macroscopic anisotropy of polycrystals. Several estimates, classical in the case of biphasic materials, have been derived for polycrystals, and can lead to significantly different predictions for the macroscopic behavior.

However, the proposed approach is restricted to linear magnetic behavior. The case of nonlinear properties is much more complicated. Nevertheless, the resolution of nonlinear homogenization problems generally rely on the resolution of a linear comparison problem (see for example [20]). In that context, the knowledge of solutions for linear homogenization problems is essential.

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