An equivalent stress for the influence of multiaxial stress on the magnetic behavior

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A main limitation of most models describing the effect of stress on the magnetic behavior is that they are restricted to uniaxial—tensile or compressive—stress. An idea to overcome this strong limitation is to define a fictive uniaxial stress, the equivalent stress that would change the magnetic behavior in a similar manner than a multiaxial one. A new definition of such an equivalent stress is proposed based on an equivalence in terms of magnetoelastic energy. This equivalent stress is compared with former proposals and validated using experimental results carried out under biaxial mechanical loading. © *2009 American Institute of Physics*. [DOI: 10.1063/1.3068646]

I. INTRODUCTION

In most of practical electromagnetic applications, magnetic materials are submitted to multiaxial stress inherited from forming process or appearing in use. On the other hand, stress is known to change significantly the magnetic behavior of materials.[1] However, the few available models describing the effect of stress on the magnetic behavior are usually restricted to uniaxial (tensile or compressive) stress (see, for instance, Refs. 2 and 3). Some attempts have been proposed in the case of biaxial loadings.[4] The development of fully multiaxial magnetoelastic models is a promising issue,[5,6] but still leads to dissuasive computational times for engineering design applications. Another solution is to introduce the multiaxiality of stress into the classical uniaxial models through the definition of a fictive uniaxial stress, the equivalent stress that would change the magnetic behavior in a similar manner than the multiaxial one. Some authors proposed such an approach in the past years.[7–9] It is shown that, in many cases, these proposals are not fully satisfactory. We propose a new definition of the equivalent stress based on an equivalence in terms of magnetoelastic energy. Assuming that a same magnetoelastic energy corresponds to a same magnetic behavior, the equivalent stress is defined as the uniaxial stress applied along the magnetic field direction that defines the same macroscopic magnetoelastic energy than the multiaxial one. A validation using experimental results carried out under biaxial mechanical loading is proposed.

II. EQUIVALENT STRESSES FROM LITERATURE

Several authors tried to define an equivalent stress for magnetoelastic behavior, usually thanks to energetic considerations and experimental observations of magnetic behavior of materials submitted to biaxial mechanical loading. Kashiwaya[7] (K) proposed the following definition for the equivalent stress σ_{eq}^{K}:

$$\sigma_{eq}^{K} = K(\sigma_1 - \sigma_{max}), \tag{1}$$

where K is a constant, σ_1 the eigenstress aligned with the magnetic field direction, and σ_{max} the maximal value of the stress tensor eigenvalues. This equivalent stress is always negative or null. Isovalues are parallel lines. If the magnetic field is applied along the direction of the maximum eigenstress, the equivalent stress is zero, so that a tensile stress or an equibiaxial tension or compression is supposed to have no effect on the magnetic behavior.

Schneider and Richardson[8] (SR) proposed the following definition for the equivalent stress σ_{eq}^{SR}:

$$\sigma_{eq}^{SR} = \sigma_1 - \sigma_2, \tag{2}$$

where σ_1 and σ_2 are the eigenstresses in the sheet plane, the magnetic loading being aligned in the direction of σ_1. The main difference with the K definition is that the area of the stress plane where $\sigma_1 > 0$ and $\sigma_2 < 0$ defines a positive equivalent stress. However an equibiaxial stress is still supposed to have no effect on the magnetic behavior.

Sablik *et al.*[9] (S) proposed the following definition for the equivalent stress σ_{eq}^{S} based on previous magnetomechanical measurements by Langman:[10]

$$\sigma_{eq}^{S} = \tfrac{1}{3}(2\sigma_1 - \sigma_2) \quad \text{for} \quad \sigma_1 < 0,$$

$$\sigma_{eq}^{S} = \tfrac{1}{3}(\sigma_1 - 2\sigma_2) \quad \text{for} \quad \sigma_1 \geq 0, \tag{3}$$

Where σ_1 is still the stress aligned with the magnetic field. Equibitraction and equibicompression do not lead to the same result, which is a significant difference with the K and SR approaches. However this model is discontinuous for $\sigma_1 = 0$ and the equivalent stress is not equal to the applied stress in the case of uniaxial loading.

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Pearson *et al.*[11] also proposed an equivalent stress for a biaxial mechanical loading. In its simplest form this equivalent stress corresponds to the SR proposal. The more refined form is a polynomial interpolation that reveals complicated to use because the parameter identification is sample dependent.

All these proposals for an equivalent stress exhibit strong limitations: the mechanical loading is restricted to biaxial stress and the magnetic field is necessarily applied along an eigendirection of the stress tensor. The definition of a more general equivalent stress is requisite considering the much more complex range of combined magnetic and mechanical loadings that can be encountered in practical applications.

III. A NEW PROPOSAL

We propose a definition for the equivalent stress σ_{eq} simply based on an equivalence in magnetoelastic energy. The magnetoelastic energy W_σ over a volume V is usually defined as follows:[12]

$$W_\sigma = \frac{1}{V}\int_V -\boldsymbol{\sigma}:\boldsymbol{\varepsilon}^\mu dV, \tag{4}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}^\mu$ are the stress and magnetostriction strain tensors. Assuming that the stress is uniform over the volume of the material (meaning that elastic incompatibilities are neglected), $\boldsymbol{\sigma}$ can be replaced by $\overline{\boldsymbol{\sigma}}$, and Eq. (4) is rewritten as

$$W_\sigma = -\overline{\boldsymbol{\sigma}}:\frac{1}{V}\int_V \boldsymbol{\varepsilon}^\mu dV = -\overline{\boldsymbol{\sigma}}:\overline{\boldsymbol{\varepsilon}}^\mu. \tag{5}$$

$\overline{\boldsymbol{\sigma}}$ and $\overline{\boldsymbol{\varepsilon}}^\mu$ are the macroscopic stress and magnetostriction strain tensors defined by Eq. (6) in a coordinate system for which direction 1 is the direction of the magnetic field. For the sake of simplicity, $\overline{\boldsymbol{\varepsilon}}^\mu$ is written for an isotropic material (so that $\overline{\varepsilon}^\mu_{22}=\overline{\varepsilon}^\mu_{33}$). An anisotropic magnetostriction would lead to a slightly different definition for the equivalent stress not detailed herein. We also assume isovolumetric magnetostriction (volume magnetostriction is neglected, so that the trace of $\overline{\boldsymbol{\varepsilon}}^\mu$ is equal to zero),

$$\overline{\boldsymbol{\sigma}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \quad \overline{\boldsymbol{\varepsilon}}^\mu = \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}. \tag{6}$$

λ is the magnetostriction strain measured in direction 1. The magnetoelastic energy can be developed as

$$W_\sigma = -\lambda\left[\sigma_{11} - \tfrac{1}{2}(\sigma_{22}+\sigma_{33})\right] = -\lambda\left[\tfrac{3}{2}\sigma_{11} - \tfrac{1}{2}\mathrm{tr}(\boldsymbol{\sigma})\right], \tag{7}$$

where $\mathrm{tr}(\boldsymbol{\sigma})$ is the trace of the stress tensor. In order to get a definition independent from the chosen coordinate system, the stress component in the direction of the magnetic field is written as $\sigma_{11} = {}^t\vec{h}\boldsymbol{\sigma}\vec{h}$. The expression for the magnetoelastic energy is finally written, for any stress tensor $\boldsymbol{\sigma}$,

$$W_\sigma = -\lambda\left[\tfrac{3}{2}\,{}^t\vec{h}\boldsymbol{\sigma}\vec{h} - \tfrac{1}{2}\mathrm{tr}(\boldsymbol{\sigma})\right]. \tag{8}$$

Let consider now a uniaxial stress $\boldsymbol{\sigma}_u$ applied in the direction 1 parallel to the magnetic field ($\boldsymbol{\sigma}_u$ is defined by $\sigma_{ij}=0$ except $\sigma_{11}=\sigma_u$). The corresponding magnetoelastic energy, according to Eq. (8), is then

$$W_\sigma = -\lambda\sigma_u. \tag{9}$$

If we assume that a same magnetoelastic energy leads to a same magnetic behavior (neglecting the effect of stress on the other energetic terms) and that the magnetostriction does not depend on stress (even though very common in magnetoelastic models, this assumption neglects a part of the effect of stress on magnetic domain distribution and consequently on magnetostatic energy), Eqs. (8) and (9) can be considered equivalent. The following expression for the equivalent stress σ_{eq} is obtained:

$$\sigma_{eq} = \tfrac{3}{2}\,{}^t\vec{h}\boldsymbol{\sigma}\vec{h} - \tfrac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}) = \tfrac{3}{2}\,{}^t\vec{h}\boldsymbol{s}\vec{h}, \tag{10}$$

where $\boldsymbol{s}$ is the deviatoric part of the stress tensor $\boldsymbol{\sigma}$. σ_{eq} is the uniaxial mechanical loading, applied in the direction $\vec{h}$ parallel to the magnetic field, which defines the same magnetoelastic energy W_σ than the multiaxial mechanical loading $\boldsymbol{\sigma}$. Several properties can be highlighted:

(i) in the case of an uniaxial stress applied in the direction of the magnetic field, the equivalent stress is the applied stress;

(ii) the definition can be applied to a fully multiaxial mechanical loading (not only biaxial);

(iii) any orientation of the stress tensor with respect to the magnetic field can be considered; and

(iv) a hydrostatic pressure leads to an equivalent stress equal to zero, in agreement with the noneffect of hydrostatic pressure on magnetic behavior.

IV. EXPERIMENTAL VALIDATION

Experiments have been performed on iron-cobalt laminations.[13] They consist in anhysteretic magnetic measurements carried out under biaxial mechanical stress in homogeneous magnetic and mechanical conditions. Seventeen biaxial (σ_1,σ_2) stress points have been tested, for stress levels varying from -60 to $+60$ MPa. The magnetic field is applied along direction 1. Figure 1 shows the secant susceptibility $\chi(\sigma_1,\sigma_2)=M(\sigma_1,\sigma_2)/H$ in the stress plane. Tension $(\sigma_1>0,\ \sigma_2=0)$ slightly increases the susceptibility and compression $(\sigma_1<0,\ \sigma_2=0)$ strongly decreases it. A uniaxial stress in the direction orthogonal to the field $(\sigma_1=0,\ \sigma_2\neq 0)$ deteriorates the magnetic behavior with a stronger effect in tension. Equibiaxial stress $(\sigma_1=\sigma_2)$ and shear stress $(\sigma_1=-\sigma_2)$ strongly deteriorate the magnetic behavior when σ_1 is negative and have a much lower effect when σ_1 is positive.

The expected susceptibility according respectively to K, SR, S, and the proposed criterion have been estimated (the experimental data for the susceptibility under uniaxial mechanical loading have been extracted from the measurements with $\sigma_2=0$). The experimental conditions correspond to biaxial stress (σ_1,σ_2) with the magnetic field along eigendirec-

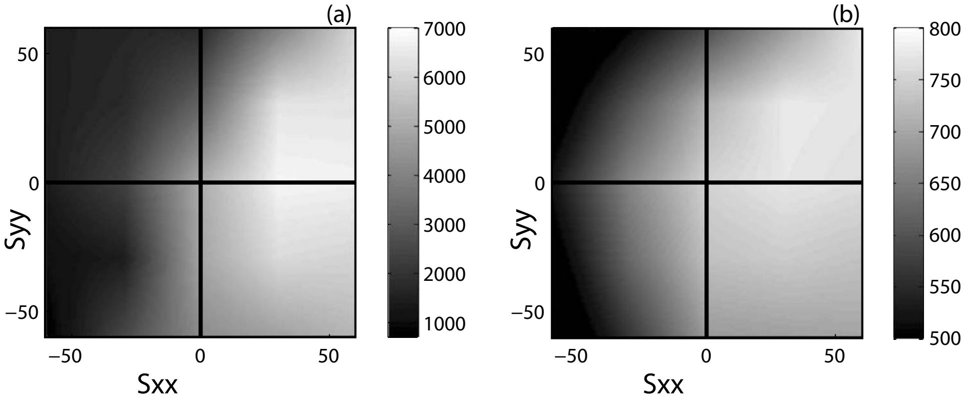

FIG. 1. Experimental secant susceptibility under mechanical loadings: (a) $H=250$ A/m and (b) $H=2500$ A/m.

tion 1. The proposed equivalent stress is then defined by $\sigma_{eq}=\sigma_1-(1/2)\sigma_2$. Figure 2 shows a map of the relative error between predicted χ_p and measured χ_e susceptibilities $(e=100\times|\chi_p-\chi_e|/\chi_e)$.

The present equivalent stress significantly reduces the relative error compared to the previous proposals. However, the errors observed in equibicompression are still not acceptable, with a level of 50% in the case of an equibicompression (−60, −60 MPa) for $H=2500$ A/m, to be compared with 70% for K and SR, and 57% for S, and a level of 238% for $H=250$ A/m, to be compared to 635% for K and SR, and 371% for S.

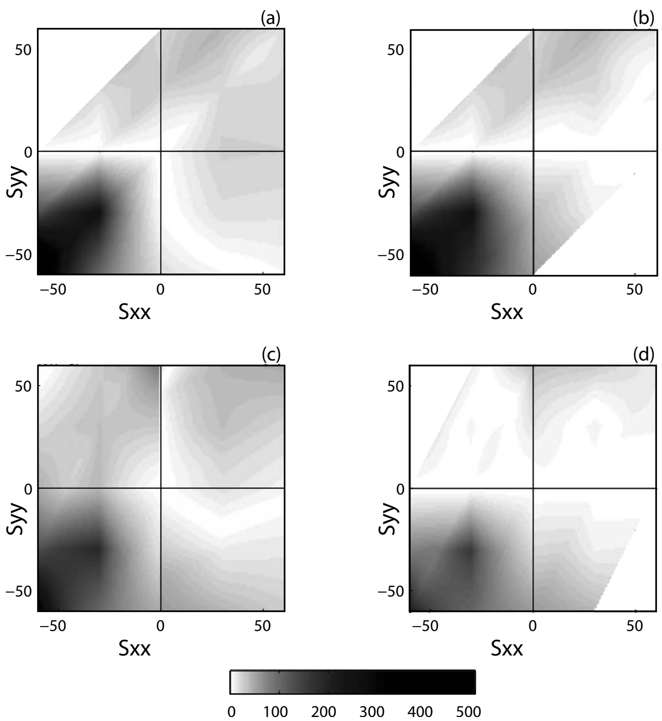

FIG. 2. Relative error (percent) for the predicted susceptibility ($H=250$ A/m) according to several equivalent stress proposals: (a) K ($K=1$), (b) SR, (c) S, and (d) the present proposal.

V. CONCLUSION

A new equivalent stress for magnetomechanical behavior is proposed. It is defined as the uniaxial mechanical loading, applied in the direction parallel to the applied magnetic field, that induces the same effect on the magnetic behavior than the corresponding multiaxial stress. Simplifying assumptions on the magnetoelastic energy leads to a simple definition of this equivalent stress. Comparisons to experimental results obtained under biaxial loadings show that the proposed equivalent stress gives more accurate predictions than the previous proposals. This equivalent stress has the heavy advantage not to be restricted to biaxial mechanical loadings but to apply to fully multiaxial configurations. Moreover it does not require any assumption on the magnetic field direction. However this new equivalent stress insufficiently describes the effect of bicompression. Improvements will consist in a better description of the effect of stress on the magnetic domain configuration. The introduction of such an effect is a work in progress and will be the object of a further communication.

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